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Όπως είναι γνωστόν, το Πανελλήνιο Συνέδριο Μαθηματικής Ανάλυσης είναι ένας θεσμός με 20ετή ιστορία και διοργανώνεται κάθε 1-2 χρόνια από τους Τομείς Μαθηματικής Ανάλυσης των ΑΕΙ Ελλάδος και Κύπρου. Το

13° Πανελλήνιο Συνέδριο Μαθηματικής Ανάλυσης

διοργανώθηκε από τον Τομέα Μαθηματικής Ανάλυσης του Τμήματος Μαθηματικών του Πανεπιστημίου Ιωαννίνων και διεξήχθη στο Πανεπιστήμιο Ιωαννίνων το διήμερο 28 και 29 Μαΐου 2010. Στο Συνέδριο αυτό συμμετείχαν πάνω από 100 Σύνεδροι που παρουσίασαν 62 εργασίες. Επίσης το παρακολούθησαν 70 προπτυχιακοί και μεταπτυχιακοί φοιτητές. Στον παρόντα Τόμο περιέχονται οι εργασίες που έστειλαν για δημοσίευση οι Σύνεδροι.

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held by the Section of Mathematical Analysis, Department of Mathematics, University of Ioannina, May 28-29, 2010. There were over 100 participants who presented 62 papers. Also 70 undergraduate and postgraduate students attended the Conference.

ΠΑΝΕΠΙΣΤΗΜΙΟ ΙΩΑΝΝΙΝΩΝ UNIVERSITY OF IOANNINA

ΤΜΗΜΑ ΜΑΘΗΜΑΤΙΚΩΝ DEPARTMENT OF MATHEMATICS

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Nabla Fractional Calculus on Time Scales and Inequalities

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Abstract

Here we develop the Nabla Fractional Calculus on Time Scales. Then we produce related integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. Finally we give inequalities applications on the time scales $\mathbb{R}$, $\mathbb{Z}$.

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1 Background and Foundation Results

For the basics on time scales we follow [1], [2], [3], [4], [9], [11], [13], [6], [7], [10].

By [15], p. 256, for $\mu, \nu > 0$ we have that

$$\int_t^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \frac{(s-t)^{\nu-1}}{\Gamma(\nu)} ds = \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \tag{1}$$

where Γ is the gamma function.

Here we consider time scales T such that $T_k = T$.

Consider the coordinatewise ld-continuous functions $\widehat{h}_\alpha : T \times T \to \mathbb{R}$, $\alpha \geq 0$, such that $\widehat{h}_0(t,s) = 1$,

$$\widehat{h}_{\alpha+1}(t,s) = \int_s^t \widehat{h}_\alpha(\tau,s)\,\nabla\tau, \tag{2}$$

$\forall\ s,t \in T$.

Here ρ is the backward jump operator and $\nu(t) = t - \rho(t)$.

Furthermore for $\alpha, \beta > 1$ we assume that

$$\int_{\rho(u)}^t \widehat{h}_{\alpha-1}(t,\rho(\tau))\,\widehat{h}_{\beta-1}(\tau,\rho(u))\,\nabla\tau = \widehat{h}_{\alpha+\beta-1}(t,\rho(u)), \tag{3}$$

valid for all $u, t \in T : u \leq t$.

In the case of $T = \mathbb{R}$; then $\rho(t) = t$, and $\widehat{h}_k(t,s) = \frac{(t-s)^k}{k!}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and define

$$\widehat{h}_\alpha(t,s) = \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

Notice that

$$\int_s^t \frac{(\tau-s)^\alpha}{\Gamma(\alpha+1)}d\tau = \frac{(t-s)^{\alpha+1}}{\Gamma(\alpha+2)} = \widehat{h}_{\alpha+1}(t,s),$$

fulfilling (2).

Furthermore we observe that $(\alpha, \beta > 1)$

$$\int_u^t \widehat{h}_{\alpha-1}(t,\tau)\,\widehat{h}_{\beta-1}(\tau,u)\,d\tau = \int_u^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{(\tau-u)^{\beta-1}}{\Gamma(\beta)}d\tau$$

$$\overset{\text{(by (1))}}{=} \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = \widehat{h}_{\alpha+\beta-1}(t,u),$$

fulfilling (3).

By Theorem 2.2 of [14], we have for $k, m \in \mathbb{N}_0$ that

$$\int_{t_0}^t \widehat{h}_k(t,\rho(\tau))\,\widehat{h}_m(\tau,t_0)\,\nabla\tau = \widehat{h}_{k+m+1}(t,t_0). \tag{4}$$

Let $T = \mathbb{Z}$, then $\rho(t) = t-1$, $t \in \mathbb{Z}$. Define $t^{\overline{0}} := 1$, $t^{\overline{k}} := t(t+1)\ldots(t+k-1)$, $k \in \mathbb{N}$, and by (2) we have $\widehat{h}_k(t,s) = \frac{(t-s)^{\overline{k}}}{k!}$, $s,t \in \mathbb{Z}$, $k \in \mathbb{N}_0$.

Here $\int_{t_0}^{t} \nabla\tau = \sum_{t_0+1}^{t}$.

Therefore by (4) we get

$$\sum_{\tau=t_0+1}^{t} \frac{(t-\tau+1)^{\overline{k}}}{k!} \frac{(\tau-t_0)^{\overline{m}}}{m!} = \frac{(t-t_0)^{\overline{k+m+1}}}{(k+m+1)!},$$

which results into

$$\sum_{\tau=t_0}^{t} \frac{(t-\tau+1)^{\overline{k-1}}}{(k-1)!} \frac{(\tau-t_0+1)^{\overline{m-1}}}{(m-1)!} = \frac{(t-t_0+1)^{\overline{k+m-1}}}{(k+m-1)!}, \tag{5}$$

confirming (3).

Next we follow [5].

Let $a, \alpha \in \mathbb{R}$, define $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$, $t \in \mathbb{R} - \{\ldots,-2,-1,0\}$, $N_a = \{a, a\pm1, a\pm 2, \ldots\}$, notice $N_0 = \mathbb{Z}$, $0^{\overline{\alpha}} = 0$, $t^{\overline{0}} = 1$, and $f : N_a \to \mathbb{R}$. Here $\rho(s) = s-1$, $\sigma(s) = s+1$, $\nu(t) = 1$. Also define

$$\nabla_a^{-n} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{n-1}}}{(n-1)!} f(s), \quad n \in \mathbb{N},$$

and in general

$$\nabla_a^{-\nu} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s),$$

where $\nu \in \mathbb{R} - \{\ldots,-2,-1,0\}$.

Here we set

$$\widehat{h}_\alpha(t,s) = \frac{(t-s)^{\overline{\alpha}}}{\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

We need

Lemma 1 *Let $\alpha > -1$, $x > \alpha+1$. Then*

$$\frac{\Gamma(x)}{\Gamma(x-\alpha)} = \frac{1}{(\alpha+1)} \left(\frac{\Gamma(x+1)}{\Gamma(x-\alpha)} - \frac{\Gamma(x)}{\Gamma(x-\alpha-1)} \right).$$

Proposition 2 *Let $\alpha > -1$. It holds*

$$\int_s^t \frac{(\tau - s)^{\overline{\alpha}}}{\Gamma(\alpha+1)} \nabla \tau = \frac{(t-s)^{\overline{\alpha+1}}}{\Gamma(\alpha+2)}, \quad t \geq s.$$

That is $\widehat{h}_\alpha$, $\alpha \geq 0$, on N_a confirm (2).

Next for $\mu, \nu > 1$, $\tau < t$, from the proof of Theorem 2.1 ([5]) we get that

$$\sum_{s=\tau}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} \frac{(s-\rho(\tau))^{\overline{\mu-1}}}{\Gamma(\mu)} = \frac{(t-\rho(\tau))^{\overline{\nu+\mu-1}}}{\Gamma(\mu+\nu)},$$

where $\tau \in \{a, ..., t\}$.

So for $t, t_0 \in N_a$ with $t_0 < t$ we obtain

$$\sum_{\tau=t_0}^{t} \frac{(t-\tau+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \frac{(\tau-t_0+1)^{\overline{\mu-1}}}{\Gamma(\mu)} = \frac{(t-t_0+1)^{\overline{\nu+\mu-1}}}{\Gamma(\mu+\nu)}, \tag{6}$$

that is confirming (3) fractionally on the time scale $T = N_a$.

Notice also here that

$$\int_a^b f(t) \nabla t = \sum_{t=a+1}^{b} f(t).$$

So fractional conditions (2) and (3) are very natural and common on time scales.

For $\alpha \geq 1$ we define the time scale ∇-Riemann-Liouville type fractional integral ($a, b \in T$)

$$J_a^\alpha f(t) = \int_a^t \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau) \nabla \tau, \tag{7}$$

(by [8] the last integral is on $(a, t] \cap T$)

$$J_a^0 f(t) = f(t),$$

where $f \in L_1([a, b] \cap T)$ (Lebesgue ∇-integrable functions on $[a, b] \cap T$, see [6], [7], [10]), $t \in [a, b] \cap T$.

Notice $J_a^1 f(t) = \int_a^t f(\tau) \nabla \tau$ is absolutely continuous in $t \in [a, b] \cap T$, see [8].

Lemma 3 *Let $\alpha > 1$, $f \in L_1([a,b] \cap T)$. Assume that $\widehat{h}_{\alpha-1}(s, \rho(t))$ is Lebesgue ∇-measurable on $([a,b] \cap T)^2$; $a, b \in T$. Then $J_a^\alpha f \in L_1([a,b] \cap T)$.*

For $u \leq t$; $u, t \in T$, we define

$$\varepsilon(t,u) = \int_{\rho(u)}^{u} \widehat{h}_{\alpha-1}(t, \rho(\tau)) \widehat{h}_{\beta-1}(\tau, \rho(u)) \nabla\tau$$
$$= \nu(u) \widehat{h}_{\alpha-1}(t, \rho(u)) \widehat{h}_{\beta-1}(u, \rho(u)), \tag{8}$$

where $\alpha, \beta > 1$.

Next we notice for $\alpha, \beta > 1$; $a, b \in T$, $f \in L_1([a,b] \cap T)$, and $\widehat{h}_{\alpha-1}(s, \rho(t))$ is continuous on $([a,b] \cap T)^2$ for any $\alpha > 1$, that

$$J_a^\alpha J_a^\beta f(t) = \int_a^t \widehat{h}_{\alpha-1}(t, \rho(\tau)) \nabla\tau \int_a^\tau \widehat{h}_{\beta-1}(\tau, \rho(u)) f(u) \nabla u.$$

Hence

$$J_a^\alpha J_a^\beta f(t) + \int_a^t f(u) \varepsilon(t,u) \nabla u = J_a^{\alpha+\beta} f(t), \quad \forall\, t \in [a,b] \cap T.$$

So we have the semigroup property

$$J_a^\alpha J_a^\beta f(t) + \int_a^t f(u) \nu(u) \widehat{h}_{\alpha-1}(t, \rho(u)) \widehat{h}_{\beta-1}(u, \rho(u)) \nabla u = J_a^{\alpha+\beta} f(t), \tag{9}$$

$\forall\, t \in [a,b] \cap T$, with $a, b \in T$.

We call the Lebesgue ∇-integral

$$D(f, \alpha, \beta, T, t) = \int_a^t f(u) \nu(u) \widehat{h}_{\alpha-1}(t, \rho(u)) \widehat{h}_{\beta-1}(u, \rho(u)) \nabla u, \tag{10}$$

$t \in [a,b] \cap T$; $a, b \in T$, the backward graininess deviation functional of $f \in L_1([a,b] \cap T)$.

If $T = \mathbb{R}$, then $D(f, \alpha, \beta, \mathbb{R}, t) = 0$.

Putting things together we have

Theorem 4 *Let $T_k = T$, $a, b \in T$, $f \in L_1([a,b] \cap T)$; $\alpha, \beta > 1$; $\widehat{h}_{\alpha-1}(s, \rho(t))$ is continuous on $([a,b] \cap T)^2$ for any $\alpha > 1$. Then*

$$J_a^\alpha J_a^\beta f(t) + D(f, \alpha, \beta, T, t) = J_a^{\alpha+\beta} f(t), \tag{11}$$

$\forall\, t \in [a,b] \cap T$.

We make

Remark 5 *Let $\mu > 2$ such that $m-1 < \mu < m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$ (ceiling of the number), $\widetilde{\nu} = m - \mu$ $(0 < \widetilde{\nu} < 1)$.*

Let $f \in C_{ld}^{m}([a,b] \cap T)$. Clearly here ([10]) f^{∇^m} is a Lebesgue ∇-integrable function.

We define the nabla fractional derivative on time scale T of order $\mu - 1$ as follows:

$$\nabla_{a*}^{\mu-1} f(t) = \left(J_a^{\widetilde{\nu}+1} f^{\nabla^m}\right)(t) = \int_a^t \widehat{h}_{\widetilde{\nu}}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla \tau, \tag{12}$$

$\forall\, t \in [a,b] \cap T$.

Notice here that $\nabla_{a*}^{\mu-1} f \in C([a,b] \cap T)$ by a simple argument using dominated convergence theorem in Lebesgue ∇-sense.

If $\mu = m$, then $\widetilde{\nu} = 0$ and by (12) we get

$$\nabla_{a*}^{m-1} f(t) = J_a^1 f^{\nabla^m}(t) = f^{\nabla^{m-1}}(t). \tag{13}$$

More generally, by [8], given that $f^{\nabla^{m-1}}$ is everywhere finite and absolutely continuous on $[a,b] \cap T$, then f^{∇^m} exists ∇-a.e. and is Lebesgue ∇-integrable on $(a,t] \cap T$, $\forall\, t \in [a,b] \cap T$, and one can plug it into (12).

We have

Theorem 6 *Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{ld}^{m}([a,b] \cap T)$, $a, b \in T$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ to be continuous on $([a,b] \cap T)^2$.*

Then

$$\int_a^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla \tau = \tag{14}$$

$$\int_a^t f^{\nabla^m}(u)\, \nu(u)\, \widehat{h}_{\mu-2}(t, \rho(u))\, \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u + \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla \tau,$$

$\forall\, t \in [a,b] \cap T$.

We need the nabla time scales Taylor formula

Theorem 7 *([2]) Let $f \in C_{ld}^{m}(T)$, $m \in \mathbb{N}$, $T_k = T$; $a, b \in T$. Then*

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t,a) f^{\nabla^k}(a) + \int_a^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau, \qquad (15)$$

$\forall\, t \in [a,b] \cap T$.

Next we present the fractional time scales nabla Taylor formula

Theorem 8 *Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{ld}^{m}(T)$, $a, b \in T$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ to be continuous on $([a,b] \cap T)^2$. Then*

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t,a) f^{\nabla^k}(a) + \qquad (16)$$

$$\int_a^t f^{\nabla^m}(u) \nu(u) \widehat{h}_{\mu-2}(t, \rho(u)) \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u + \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla\tau,$$

$\forall\, t \in [a,b] \cap T$.

Corollary 9 *All as in Theorem 8. Additionally suppose $f^{\nabla^k}(a) = 0$, $k = 0, 1, ..., m-1$. Then*

$$A(t) := f(t) - D\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T, t\right) \qquad (17)$$

$$= f(t) - \int_a^t f^{\nabla^m}(u) \nu(u) \widehat{h}_{\mu-2}(t, \rho(u)) \widehat{h}_{\widetilde{\nu}}(u, \rho(u)) \nabla u$$

$$= \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla\tau,$$

$\forall\, t \in [a,b] \cap T$.

Notice here that $D\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T, t\right) \in C_{ld}([a,b] \cap T)$. Also the R.H.S (17) is a continuous function in $t \in [a,b] \cap T$.

2 Fractional Nabla Inequalities on Time Scales

We present a Poincaré type related inequality.

Theorem 10 *Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{ld}^{m}(T)$, $a, b \in T$, $a \le b$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ to be continuous on $([a,b] \cap T)^2$, and $f^{\nabla^k}(a) = 0$, $k = 0, 1, ..., m-1$. Here $A(t) = f(t) - D\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu}+1, T, t\right)$, $t \in [a,b] \cap T$; and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.*

Then

$$\int_a^b |A(t)|^q \nabla t \le \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right)^{\frac{q}{p}} \nabla t \right) \left(\int_a^b \left| \nabla_{a*}^{\mu-1} f(t) \right|^q \nabla t \right). \tag{18}$$

Next we give a related Sobolev inequality.

Theorem 11 *Here all as in Theorem 10. Let $r \ge 1$ and denote*

$$\|f\|_r = \left(\int_a^b |f(t)|^r \nabla t \right)^{\frac{1}{r}}. \tag{19}$$

Then

$$\|A\|_r \le \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right)^{\frac{r}{p}} \nabla t \right)^{\frac{1}{r}} \left\| \nabla_{a*}^{\mu-1} f \right\|_q. \tag{20}$$

Next we give an Opial type related inequality.

Theorem 12 *Here all as in Theorem 10. Additionally assume that*

$$\left| \nabla_{a*}^{\mu-1} f \right| \text{ is increasing on } [a,b] \cap T. \tag{21}$$

Then

$$\int_a^b |A(t)| \left| \nabla_{a*}^{\mu-1} f(t) \right| \nabla t \le$$

$$(b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right) \nabla t \right)^{\frac{1}{p}} \left(\int_a^b \left(\nabla_{a*}^{\mu-1} f(t) \right)^{2q} \nabla t \right)^{\frac{1}{q}}. \tag{22}$$

It follows related Ostrowski type inequalities.

Theorem 13 *Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f \in C_{ld}^{m}(T)$, $a, b \in T$, $a \leq b$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\widetilde{\nu}}(s, \rho(t))$ to be continuous on $([a,b] \cap T)^2$, and $f^{\nabla^k}(a) = 0$, $k = 1, ..., m-1$. Denote $A(t) = f(t) - D\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T, t\right)$, $t \in [a,b] \cap T$.*

Then

$$\left| \frac{1}{b-a} \int_a^b A(t) \nabla t - f(a) \right| \leq$$

$$\frac{1}{b-a} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right| \nabla \tau \right) \nabla t \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{\infty, [a,b] \cap T}. \tag{23}$$

Theorem 14 *All as in Theorem 13. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{1}{b-a} \int_a^b A(t) \nabla t - f(a) \right| \leq$$

$$\frac{1}{b-a} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right)^{\frac{1}{p}} \nabla t \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{q, [a,b] \cap T}. \tag{24}$$

We finish general fractional nabla time scales inequalities with a related Hilbert-Pachpatte type inequality.

Theorem 15 *Let $\varepsilon > 0$, $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$; $f_i \in C_{ld}^{m}(T_i)$, $a_i, b_i \in T_i$, $a_i \leq b_i$, $T_{ik} = T_i$ time scale, $i = 1, 2$. Suppose $\widehat{h}_{\mu-2}^{(i)}(s_i, \rho_i(t_i))$, $\widehat{h}_{\widetilde{\nu}}^{(i)}(s_i, \rho_i(t_i))$ to be continuous on $([a_i, b_i] \cap T_i)^2$, and $f_i^{\nabla^k}(a_i) = 0$, $k = 0, 1, ..., m-1$; $i = 1, 2$. Here $A_i(t_i) = f_i(t_i) - D_i(f_i^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, T_i, t_i)$, $t_i \in [a_i, b_i] \cap T_i$; $i = 1, 2$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.*

Call

$$F(t_1) = \int_{a_1}^{t_1} \left(\left| \widehat{h}_{\mu-2}^{(1)}(t_1, \rho_1(\tau_1)) \right| \right)^p \nabla \tau_1,$$

for all $t_1 \in [a_1, b_1]$, and

$$G(t_2) = \int_{a_2}^{t_2} \left(\left| \widehat{h}_{\mu-2}^{(2)}(t_2, \rho_2(\tau_2)) \right| \right)^q \nabla \tau_2,$$

for all $t_2 \in [a_2, b_2]$ *(where* $\widehat{h}^{(i)}_{\mu-2}$*,* ρ_i *are the corresponding* $\widehat{h}_{\mu-2}$*,* ρ *to* T_i*,* $i = 1, 2$*).*

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|A_1(t_1)| |A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \nabla t_1 \nabla t_2 \leq$$

$$(b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left| \nabla^{\mu-1}_{a_1 *} f_1(t_1) \right|^q \nabla t_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left| \nabla^{\mu-1}_{a_2 *} f_2(t_2) \right|^p \nabla t_2 \right)^{\frac{1}{p}}. \tag{25}$$

(above double time scales Riemann nabla integration is considered in the natural interative way).

3 Applications

I) Here $T = \mathbb{R}$ case.

Let $\mu > 2$ such that $m - 1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$, $f \in C^m([a, b])$, $a, b \in \mathbb{R}$.

The nabla fractional derivative on $\mathbb{R}$ of order $\mu - 1$ is defined as follows:

$$\nabla^{\mu-1}_{a*} f(t) = \left(J^{\widetilde{\nu}+1}_a f^{(m)} \right)(t) = \frac{1}{\Gamma(\widetilde{\nu} + 1)} \int_a^t (t - \tau)^{\widetilde{\nu}} f^{(m)}(\tau) \, d\tau, \tag{26}$$

$\forall \, t \in [a, b]$.

Notice that $\nabla^{\mu-1}_{a*} f \in C([a, b])$, and $A(t) = f(t)$, $\forall \, t \in [a, b]$.

We give a Poincaré type inequality.

Theorem 16 *Let* $\mu > 2$*,* $m - 1 < \mu < m \in \mathbb{N}$*,* $f \in C^m(\mathbb{R})$*,* $a, b \in \mathbb{R}$*,* $a \leq b$*. Suppose* $f^{(k)}(a) = 0$*,* $k = 0, 1, ..., m - 1$*. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_a^b |f(t)|^q \, dt \leq \frac{(b - a)^{(\mu-1)q}}{(\Gamma(\mu - 1))^q (\mu - 1) q ((\mu - 2) p + 1)^{q-1}} \left(\int_a^b \left| \nabla^{\mu-1}_{a*} f(t) \right|^q dt \right). \tag{27}$$

Proof. By Theorem 10. ■

We give a Sobolev type inequality.

Theorem 17 *All as in Theorem 16. Let $r \geq 1$. Then*

$$\|f\|_r \leq \frac{(b-a)^{\mu-2+\frac{1}{p}+\frac{1}{r}}}{\Gamma(\mu-1)\left((\mu-2)p+1\right)^{\frac{1}{p}}\left((\mu-2)r+\frac{r}{p}+1\right)^{\frac{1}{r}}}\left\|\nabla_{a*}^{\mu-1}f\right\|_q. \quad (28)$$

Proof. By Theorem 11. ■

We continue with an Opial type inequality.

Theorem 18 *All as in Theorem 16. Assume $\left|\nabla_{a*}^{\mu-1}f\right|$ is increasing on $[a,b]$.*

$$\int_a^b |f(t)|\left|\nabla_{a*}^{\mu-1}f(t)\right| dt \leq$$

$$\frac{(b-a)^{\mu-\frac{1}{q}}}{\Gamma(\mu-1)\left[\left((\mu-2)p+1\right)\left((\mu-2)p+2\right)\right]^{\frac{1}{p}}}\left(\int_a^b \left(\nabla_{a*}^{\mu-1}f(t)\right)^{2q} dt\right)^{\frac{1}{q}}. \quad (29)$$

Proof. By Theorem 12. ■

Some Ostrowski type inequalities follow.

Theorem 19 *Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $f \in C^m(\mathbb{R})$, $a, b \in \mathbb{R}$, $a \leq b$. Suppose $f^{(k)}(a) = 0$, $k = 1, ..., m-1$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right| \leq \frac{(b-a)^{\mu-1}}{\Gamma(\mu+1)}\left\|\nabla_{a*}^{\mu-1}f\right\|_{\infty,[a,b]}. \quad (30)$$

Proof. By Theorem 13. ■

Theorem 20 *Here all as in Theorem 19. Let $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right| \leq \frac{(b-a)^{\mu-\frac{1}{q}-1}}{\Gamma(\mu-1)\left(\mu-\frac{1}{q}\right)\left((\mu-2)p+1\right)^{\frac{1}{p}}}\left\|\nabla_{a*}^{\mu-1}f\right\|_{q,[a,b]}. \quad (31)$$

Proof. By Theorem 14. ■

We finish this subsection with a Hilbert-Pachpatte inequality on $\mathbb{R}$.

Theorem 21 *Let* $\varepsilon > 0$, $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $i = 1,2$; $f_i \in C^m(\mathbb{R})$, $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$, $f_i^{(k)}(a_i) = 0$, $k = 0,1,...,m-1$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Call

$$F(t_1) = \frac{(t_1 - a_1)^{(\mu-2)p+1}}{(\Gamma(\mu-1))^p((\mu-2)p+1)},$$

$t_1 \in [a_1, b_1]$, *and*

$$G(t_2) = \frac{(t_2 - a_2)^{(\mu-2)q+1}}{(\Gamma(\mu-1))^q((\mu-2)q+1)},$$

$t_2 \in [a_2, b_2]$.

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{|f_1(t_1)||f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} dt_1 dt_2 \leq$$

$$(b_1 - a_1)(b_2 - a_2)\left(\int_{a_1}^{b_1} \left|\nabla_{a_1*}^{\mu-1} f_1(t_1)\right|^q dt_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|\nabla_{a_2*}^{\mu-1} f_2(t_2)\right|^p dt_2\right)^{\frac{1}{p}}. \tag{32}$$

Proof. By Theorem 15. ∎

II) Here $T = \mathbb{Z}$ case.

Let $\mu > 2$ such that $m-1 < \mu < m \in \mathbb{N}$, $\widetilde{\nu} = m - \mu$, $a, b \in \mathbb{Z}$, $a \leq b$. Here $f : \mathbb{Z} \to \mathbb{R}$, and $f^{\nabla^m}(t) = \nabla^m f(t) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(t-k)$.

The nabla fractional derivative on $\mathbb{Z}$ of order $\mu - 1$ is defined as follows:

$$\nabla_{a*}^{\mu-1} f(t) = \left(J_a^{\widetilde{\nu}+1}(\nabla^m f)\right)(t) = \frac{1}{\Gamma(\widetilde{\nu}+1)} \sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\widetilde{\nu}}} (\nabla^m f)(\tau), \tag{33}$$

$\forall\, t \in [a, \infty) \cap \mathbb{Z}$.

Notice here that $\nu(t) = 1$, $\forall\, t \in \mathbb{Z}$, and

$$\begin{aligned} A(t) &= f(t) - D(\nabla^m f, \mu-1, \widetilde{\nu}+1, \mathbb{Z}, t) \\ &= f(t) - \sum_{u=a+1}^{t} (\nabla^m f(u)) \frac{(t-u+1)^{\overline{\mu-2}}}{\Gamma(\mu-1)}, \end{aligned} \tag{34}$$

$\forall\, t \in [a, \infty) \cap \mathbb{Z}$.

We give a discrete fractional Poincaré type inequality.

Theorem 22 *Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $a \leq b$, $f : \mathbb{Z} \to \mathbb{R}$. Assume $\nabla^k f(a) = 0$, $k = 0, 1, ..., m-1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{t=a+1}^{b} |A(t)|^q \leq$$

$$\frac{1}{(\Gamma(\mu-1))^q} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{(\mu-2)}p} \right) \right) \left(\sum_{t=a+1}^{b} \left| \nabla_{a*}^{\mu-1} f(t) \right|^q \right). \tag{35}$$

Proof. By Theorem 10. ∎

We continue with a discrete fractional Sobolev type inequality.

Theorem 23 *Here all as in Theorem 22. Let $r \geq 1$ and denote*

$$\|f\|_r = \left(\sum_{t=a+1}^{b} |f(t)|^r \right)^{\frac{1}{r}}.$$

Then

$$\|A\|_r \leq \frac{1}{\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{(\mu-2)}p} \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \left\| \nabla_{a*}^{\mu-1} f \right\|_q. \tag{36}$$

Proof. By Theorem 11. ∎

Next we give a discrete fractional Opial type inequality.

Theorem 24 *Here all as in Theorem 22. Assume that $\left|\nabla_{a*}^{\mu-1} f\right|$ is increasing on $[a, b] \cap \mathbb{Z}$. Then*

$$\sum_{t=a+1}^{b} |A(t)| \left| \nabla_{a*}^{\mu-1} f(t) \right| \leq$$

$$\frac{(b-a)^{\frac{1}{q}}}{\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{(\mu-2)}p} \right) \right)^{\frac{1}{p}} \left(\sum_{t=a+1}^{b} \left(\nabla_{a*}^{\mu-1} f(t) \right)^{2q} \right)^{\frac{1}{q}}. \tag{37}$$

Proof. By Theorem 12. ■

It follows related discrete fractional Ostrowski type inequalities.

Theorem 25 *Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $a \leq b$, $f : \mathbb{Z} \to \mathbb{R}$. Assume $\nabla^k f(a) = 0$, $k = 1, ..., m-1$.*

Then

$$\left| \frac{1}{b-a} \sum_{t=a+1}^{b} A(t) - f(a) \right| \leq$$

$$\frac{1}{(b-a)\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{\overline{\mu-2}} \right) \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{\infty,[a,b]\cap\mathbb{Z}} . \quad (38)$$

Proof. By Theorem 13. ■

Theorem 26 *All as in Theorem 25. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{1}{b-a} \sum_{t=a+1}^{b} A(t) - f(a) \right| \leq$$

$$\frac{1}{(b-a)\Gamma(\mu-1)} \left(\sum_{t=a+1}^{b} \left(\sum_{\tau=a+1}^{t} (t-\tau+1)^{(\overline{\mu-2})p} \right)^{\frac{1}{p}} \right) \left\| \nabla_{a*}^{\mu-1} f \right\|_{q,[a,b]\cap\mathbb{Z}} .$$

(39)

Proof. By Theorem 14. ■

We finish article with a discrete fractional Hilbert-Pachpatte type inequality.

Theorem 27 *Let $\varepsilon > 0$, $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$; $i = 1, 2$; $f_i : \mathbb{Z} \to \mathbb{R}$, $a_i, b_i \in \mathbb{Z}$, $a_i \leq b_i$. Suppose $\nabla^k f_i(a_i) = 0$, $k = 0, 1, ..., m-1$. Here $A_i(t_i) = f_i(t_i) - \sum_{u_i=a_i+1}^{t_i} (\nabla^m f(u_i)) \frac{(t_i-u_i+1)^{\overline{\mu-2}}}{\Gamma(\mu-1)}$, $\forall\, t_i \in [a_i, \infty) \cap \mathbb{Z}$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.*

Call

$$F(t_1) = \sum_{\tau_1=a_1+1}^{t_1} \frac{(t_1-\tau_1+1)^{(\overline{\mu-2})p}}{(\Gamma(\mu-1))^p},$$

$\forall\ t_1 \in [a_1, \infty) \cap \mathbb{Z}$, *and*

$$G(t_2) = \sum_{\tau_2=a_2+1}^{t_2} \frac{(t_2 - \tau_2 + 1)^{\overline{(\mu-2)}q}}{(\Gamma(\mu-1))^q},$$

$\forall\ t_2 \in [a_2, \infty) \cap \mathbb{Z}$.

Then

$$\sum_{t_1=a_1+1}^{b_1} \sum_{t_2=a_2+1}^{b_2} \frac{|A_1(t_1)|\,|A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \leq$$

$$(b_1 - a_1)(b_2 - a_2)\left(\sum_{t_1=a_1+1}^{b_1} \left|\nabla_{a_1*}^{\mu-1} f_1(t_1)\right|^q\right)^{\frac{1}{q}} \left(\sum_{t_2=a_2+1}^{b_2} \left|\nabla_{a_2*}^{\mu-1} f_2(t_2)\right|^p\right)^{\frac{1}{p}}. \tag{40}$$

Proof. By Theorem 15. ■

We intend to publish the complete article with full proofs elsewhere.

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Schur and matrix theorems with respect to $\mathcal{I}$-convergence

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Abstract

Some Schur and basic matrix theorems with respect to ideal convergence are proved. Moreover some examples are given.

1 Introduction.

The theory of convergence with respect to ideals was introduced in [11] and is deeply studied in the literature, in particular in problems concerning limit and integrals. Note that, in general, ideal convergence is strictly weaker than ordinary convergence (see [11]).

Here we present some versions of Schur-type and basic matrix theorems in which the existence of the "pointwise" limit measure is required only with respect to the convergence generated by a fixed P-ideal $\mathcal{I}$. Note that the ideal $\mathcal{I}_d$ of the subsets of the natural numbers having zero asymptotic density is a P-ideal (see [11]), and that $\mathcal{I}_d$-convergence coincides with the so-called "statistical convergence" (see [9]).

We prove a Schur-type theorem with respect to the given ideal, and we show by a counterexample that the existence of the "$\mathcal{I}$-limit" measure, even when it is equal to zero, is not enough to get the classical uniform (s)-boundedness. Furthermore, we give some examples concerning the relations existing between the classical convergence and $\mathcal{I}$-convergence when the involved ideals are not maximal.

2 Ideal limit theorems for measures

For the following see also [4, 6, 7, 11, 12].

Definitions 2.1 *(a) Let $X \neq \emptyset$ be any set. By $\mathcal{P}(X)$ we denote the powerset of X.*

(b) A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an ideal of X iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$

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and $B \subseteq A$ we get $B \in \mathcal{I}$.

(c) An ideal $\mathcal{I}$ is said to be non-trivial iff $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$.

(d) A non-trivial ideal $\mathcal{I}$ is said to be admissible iff it contains all singletons.

(e) An admissible ideal $\mathcal{I}$ is called a P-ideal iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subseteq X$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

(f) Let now $\mathcal{I}$ be any fixed admissible ideal and $\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{X \setminus I : I \in \mathcal{I}\}$ be its dual filter. A sequence $(x_n)_n$ in $\mathbb{R}$ $\mathcal{I}$-converges to $x \in \mathbb{R}$ iff for all $\varepsilon > 0$, $\{n \in \mathbb{N} : |x_n - x| > \varepsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I} - \lim_n x_n = x$.

(g) A sequence $(x_n)_n$ in $\mathbb{R}$ is $\mathcal{I}$-Cauchy iff for each $\varepsilon > 0$ there exists $q \in \mathbb{N}$ such that $\{n \in \mathbb{N} : |x_n - x_q| > \varepsilon\} \in \mathcal{I}$.

(h) We then define $\mathcal{I} - \sum_{j=1}^{\infty} x_j = \mathcal{I} - \lim_n \sum_{j=1}^{n} x_j$ and $\mathcal{I} - \ell^1 = \Big\{(x_n)_n \in \mathbb{R}^{\mathbb{N}} : \mathcal{I} - \sum_{j=1}^{\infty} |a_j| \in \mathbb{R}\Big\}$.

(i) A sequence $(x_n)_n$ in $\mathbb{R}$ $\mathcal{I}^*$-converges to $x \in \mathbb{R}$ iff there exists $A \in \mathcal{F}(\mathcal{I})$ with $\lim_{n \in A} x_n = x$. Then we write $I^* - \lim_n x_n = x$.

(j) If $\mathcal{I}$ is an ideal of $\mathbb{N}^2$, then the real-valued double sequence $(x_{i,j})_{i,j}$ $\mathcal{I}$-converges to $x \in \mathbb{R}$ iff for all $\varepsilon > 0$ $\{(i,j) \in \mathbb{N}^2 : |x_{i,j} - x| > \varepsilon\} \in \mathcal{I}$. Then we write $\mathcal{I} - \lim_{i,j} x_{i,j} = x$.

(k) The double real-valued sequence $(x_{i,j})_{i,j}$ is called $\mathcal{I}$-Cauchy iff for all $\varepsilon > 0$ there exists $(p,q) \in \mathbb{N}^2$ such that $\{(i,j) \in \mathbb{N}^2 : |x_{i,j} - x_{p,q}| > \varepsilon\} \in \mathcal{I}$.

From now on let Σ be any σ-algebra and $\mathcal{I}$ any admissible ideal of $\mathbb{N}$.

(l) A finitely additive measure $m : \Sigma \to \mathbb{R}$ is said to be $\mathcal{I}$-s-bounded iff for every disjoint sequence $(H_n)_n$ in Σ we have $\mathcal{I}$-$\lim_n v(m)(H_n) = 0$, where $v(m)$ denotes the semivariation of m.

The finitely additive measures $m_j : \Sigma \to \mathbb{R}$, $j \in \mathbb{N}$, are called uniformly $\mathcal{I}$-s-bounded iff $\mathcal{I}$-$\lim_n[\sup_j v(m_j)(H_n)] = 0$, whenever $(H_n)_n$ is a sequence of pairwise disjoint elements of Σ.

(m) The finitely additive measure $m : \Sigma \to \mathbb{R}$ is said to be $\mathcal{I}$-σ-additive iff for every disjoint sequence $(H_n)_n$ in Σ we get: $\mathcal{I}$-$\lim_n v(m)\Big(\bigcup_{\ell=n}^{\infty} H_\ell\Big) = 0$.

The finitely additive measures $m_j : \Sigma \to \mathbb{R}$, $j \in \mathbb{N}$, are called uniformly $\mathcal{I}$-σ-additive iff for each disjoint sequence $(H_n)_n$ in Σ we have $\mathcal{I}$-$\lim_n \Big[\sup_j v(m_j)\Big(\bigcup_{\ell=n}^{\infty} H_\ell\Big)\Big] = 0$.

Examples 2.2 (a) Let $\mathcal{I}_{fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$. Then $\mathcal{I}_{fin}$ is an ideal of $\mathbb{N}$ and $\mathcal{I}_{fin}$-convergence coincides with the ordinary convergence.

(b) Let $A \subseteq \mathbb{N}$. We denote by $d(A) := \lim_n \frac{card(A \cap \{1,\dots,n\})}{n}$ (if this limit exists in $\mathbb{R}$) the natural or asymptotic density of A. We set $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then $\mathcal{I}_d$ is an ideal of $\mathbb{N}$ and $\mathcal{I}_d$-convergence coincides with the statistical convergence (see [8]).

(c) $\mathcal{I}_{fin}$ and $\mathcal{I}_d$ are well known cases of P-ideals (see [4, 11]).

Remarks 2.3 *(a) Since $\mathbb{R}$ with the usual metric is complete we have that a (double) sequence is*

$\mathcal{I}$-Cauchy iff it is $\mathcal{I}$-convergent (for a proof see [4, 6, 12]).

(b) By Example 2.2 (a) and Definition 2.1(l) we get that $\mathcal{I}_{fin}$-s-boundedness of a measure coincides with ordinary s-boundedness, uniform-$\mathcal{I}_{fin}$-s-boundedness of a measure sequence coincides with the ordinary uniform-s-boundedness and similarly for the σ-additivity notions (Definition 2.1(m)).

Proposition 2.4 *If $\lim_n x_n = x$, then $\mathcal{I} - \lim_n x_n = x$. Moreover, if $(x_n)_n$ is a monotone sequence in $\mathbb{R}$ and $x \in \mathbb{R}$, then $\mathcal{I} - \lim_n x_n = x$ if and only if $\lim_n x_n = x$.*

Proof: The first statement is obvious since $\mathcal{I}$ is an admissible ideal (see also [10]). We now turn to the second statement. It is enough to prove the "only if" implication. Without loss of generality, assume that $(x_n)_n$ is increasing. By hypothesis, for all $\varepsilon > 0$ there exists an integer $n^* \in \mathbb{N}$ with $0 \leq x - x_{n^*} \leq \varepsilon$.

By monotonicity we get: $0 \leq x - x_n \leq x - x_{n^*} \leq \varepsilon$ for any $n \geq n^*$. So the sequence $(x_n)_n$ converges monotonically to x. This concludes the proof. $\square$

Proposition 2.5 $\mathcal{I} - \ell^1 = \ell^1$.

Proof: Immediate from Proposition 2.4.

Proposition 2.6 *Let $\mathcal{I}$ be a P-ideal and $(x_n)_n$ be a sequence in $\mathbb{R}$, such that $\mathcal{I} - \lim_n x_n = x \in \mathbb{R}$. Then there exists a subsequence $(x_{n_q})_q$ of $(x_n)_n$, such that $\lim_q x_{n_q} = x$.*

Proof: See [11, Theorem 3.2].

Proposition 2.7 *The $\mathcal{I}^*$-convergence of sequences implies always the $\mathcal{I}$-convergence. Moreover, if $(x_n)_n$ is a sequence in $\mathbb{R}$, $\mathcal{I}$-convergent to $\xi \in \mathbb{R}$, and $\mathcal{I}$ is a P-ideal, then $(x_n)_n$ $\mathcal{I}^*$-converges to ξ.*

Proof: See [11].

We now prove the following:

Proposition 2.8 *Let $(x_{i,j})_{i,j}$ be a double sequence in $\mathbb{R}$, $\mathcal{I}$ be any P-ideal, $\mathcal{F} = \mathcal{F}(\mathcal{I})$ be its dual filter, and let us suppose that $\mathcal{I} - \lim_i x_{i,j} = x_j$ for every $j \in \mathbb{N}$.*

Then there exists $B_0 \in \mathcal{F}$ such that $\lim_{h \in B_0} x_{h,j} = x_j$ for all $j \in \mathbb{N}$.

Proof: Since $\mathcal{I}$ is a P-ideal, by virtue of Proposition 2.7 we get $\mathcal{I}^* - \lim_i x_{i,j} = x_j$ for every $j \in \mathbb{N}$. Hence there is a sequence $(A_j)_j$ in $\mathcal{F}$ such that $\lim_{i \in A_j} x_{i,j} = x_j$ for all $j \in \mathbb{N}$. As $\mathcal{I}$ is a P-ideal, there is a sequence of sets $(B_j)_j$ in $\mathcal{F}$ such that $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $B_0 := \bigcap_{j=1}^{\infty} B_j \in \mathcal{F}$. Since $\lim_{i \in A_j} x_{i,j} = x_j$ for all j, then we get also $\lim_{i \in B_j} x_{i,j} = x_j$ for all j. Let $B_0 = \{p_1 < \ldots < p_h < \ldots\}$ and choose arbitrarily $j \in \mathbb{N}$: then, since $B_0 \subset B_j$, in correspondence with ε an integer $\overline{h} = \overline{h}(j)$ can be found, with the property that $|x_{p_h,j} - x_j| \leq \varepsilon$ whenever $h > \overline{h}$. This concludes the proof. $\square$

Equivalence between (uniform) σ-additivity and (uniform) $\mathcal{I}$-σ-additivity is an immediate consequence of Proposition 2.4. We now prove this result concerning s-boundedness: an analogous one can be given for uniform s-boundedness.

Proposition 2.9 *Let $\mathcal{I}$ be a P-ideal. Then every measure m is $\mathcal{I}$-s-bounded if and only if it is s-bounded.*

First of all, note that s-boundedness implies always $\mathcal{I}$-s-boundedness, since usual convergence implies $\mathcal{I}$-convergence (see [11]). Concerning the converse implication, let $(H_n)_n$ be any disjoint sequence in $\mathcal{A}$, and pick *any* subsequence $(H_{n_s})_s$ of $(H_n)_n$. Since, by $\mathcal{I}$-s-boundedness, we have $\mathcal{I}$-$\lim_s m(H_{n_s}) = 0$, then by Proposition 2.6 there is a sub-subsequence $(H_{n_{s_k}})_k$ of $(H_{n_s})_n$ with $\lim_k m(H_{n_{s_k}}) = 0$. By property (U) of the ordinary convergence (see [11]), we get $\lim_n m(H_n) = 0$, that is the assertion. $\square$

We now give some versions of the Schur lemma with respect to the ideal convergence.

Theorem 2.10 *Let $m_i : \mathcal{P}(\mathbb{N}) \to \mathbb{R}, i \in \mathbb{N}$, be a sequence of positive σ-additive measures, $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a P-ideal. Suppose that $m_0(E) := \mathcal{I} - \lim_i m_i(E)$ exists in $\mathbb{R}$ for every $E \subset \mathbb{N}$, and that m_0 is σ-additive on $\mathcal{P}(\mathbb{N})$.*

Then there exists $A \in \mathcal{F}(\mathcal{I})$ such that $\lim_j [\sup_{i\in A} m_i(H_j)] = \inf_j [\sup_{i\in A} m_i(H_j)] = 0$, where $H_j := \{j, j+1, \ldots\}$, $j \in \mathbb{N}$.

Proof: Let $(H_j)_j$ be as in the hypotheses: note that $H_j \downarrow \emptyset$. For each $j \in \mathbb{N}$, put $C_j := \{j\}$. Since $\mathcal{I}$ is a P-ideal and we deal with only countably many "objects", there exists a set $A \in \mathcal{F}(\mathcal{I})$ with: $\lim_{i\in A} m_i(E) = m_0(E)$ for every set E belonging to the algebra $\mathcal{L}_0$ of all finite and cofinite subsets of $\mathbb{N}$; $\lim_k m_i(H_k) = 0$ for all $i \in \mathbb{N} \cup \{0\}$. We claim that each map $m_i|_{\mathcal{L}_0}$ admits a countably additive extension $\widetilde{m_i}$ to $\mathcal{P}(\mathbb{N})$, and that such extension is the unique finitely additive map, defined on $\mathcal{P}(\mathbb{N})$ and agreeing with m_i on $\mathcal{L}_0$.

For every $i \geq 0$ and $P \subset \mathbb{N}$ set

$$\widetilde{m_i}(P) := \sup_n \, m_i(\cup_{j\in P, j\leq n} C_j) = \lim_n \, m_i(\cup_{j\in P, j\leq n} C_j).$$

It is easy to check that $\widetilde{m_i}$ is an extension of m_i. Moreover, if $\widehat{m_i}$ is any other extension of m_i, we have:

$$\widehat{m_i}(P) \geq \sup_n \, m_i(\cup_{j\in P, j\leq n} C_j),$$

by monotonicity of $\widehat{m_i}$;

$$\widehat{m_i}(P) = m_i(\cup_{j\in P, j\leq n} C_j) + \widehat{m_i}(\cup_{j\in P, j>n} C_j) \leq m_i(\cup_{j\in P, j\leq n} C_j) + m_i(H_{n+1})$$

for all $n \in \mathbb{N}$. Thus we get:

$$0 \leq \widehat{m_i}(P) - m_i(\cup_{j\in P, j\leq n} C_j) \leq m_i(H_{n+1})$$

for all $n \in \mathbb{N}$. As $m_i(H_n) \downarrow 0$, we obtain

$$\widehat{m_i}(P) = \lim_n \, m_i(\cup_{j\in P, j\leq n} C_j) = \widetilde{m_i}(P),$$

which proves the claim about $\widetilde{m_i}$.

We now prove that $\widetilde{m_0}(P) = \lim_{i \in A} \widetilde{m_i}(P)$ for every $P \in \mathcal{P}(\mathbb{N})$. For each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $m_0(H_k) < \varepsilon$ whenever $k > k_0$, since the "limit" measure m_0 is positive and σ-additive. Moreover, there is $i_0 \in A$, $i_0 = i_0(\varepsilon, k_0)$ such that for any $i \in A$, $i \geq i_0$,

$$\left| m_i \left(\bigcup_{j \leq k_0, j \in P} C_j \right) - m_0 \left(\bigcup_{j \leq k_0, j \in P} C_j \right) \right| \leq \varepsilon, \quad |m_i(H_{k_0+1}) - m_0(H_{k_0+1})| \leq \varepsilon.$$

Thus for each $i \in A$, $i \geq i_0$, we get:

$$\begin{aligned} 0 \ &\leq \ |\widetilde{m_i}(P) - \widetilde{m_0}(P)| \leq \left| m_i \left(\bigcup_{j \leq k_0, j \in P} C_j \right) - m_0 \left(\bigcup_{j \leq k_0, j \in P} C_j \right) \right| + \\ &+ \ \widetilde{m_i} \left(\bigcup_{j > k_0, j \in P} C_j \right) + \widetilde{m_0} \left(\bigcup_{j > k_0, j \in P} C_j \right) \\ &\leq \ \varepsilon + |m_i(H_{k_0+1}) - m_0(H_{k_0+1})| + 2\, m_0(H_{k_0+1}) \leq 4\,\varepsilon. \end{aligned}$$

Therefore $\lim_{i \in A} \widetilde{m_i}(P) = \widetilde{m_0}(P)$ for all $P \in \mathcal{P}(\mathbb{N})$, that is the claim.

Let now $\mathbb{N} \supset A_s \downarrow \emptyset$. It is not difficult to show that there is a subsequence $(A_{s_k})_{k \geq 2}$ with $\widetilde{m_i}(A_{s_k}) \leq \widetilde{m_i}(H_k)$ for any i and $k \in \mathbb{N}$. Indeed, if there is $q_2 \in \mathbb{N}$ with $A_s \supset C_1$ for $s > q_2$, then it follows easily that $\widetilde{m_i}(C_1) = 0$ for all $i \in \mathbb{N} \cup \{0\}$. So, in this case, we get: $\widetilde{m_i}(A_s) = \widetilde{m_i}(A_s \setminus C_1) \leq \widetilde{m_i}(H_2)$ for all $s > q_2$ and $i \in \mathbb{N} \cup \{0\}$. Otherwise there is $l_2 > 1$ such that $A_{l_2} \subset H_2$, and hence $\widetilde{m_i}(A_{l_2}) \leq \widetilde{m_i}(H_2)$ for all i. In any case, for at least an index s_2 and for any i we have: $\widetilde{m_i}(A_{s_2}) \leq \widetilde{m_i}(H_2)$.

At the following step, we get still two cases. If there is a positive integer q_3 with $A_s \supset C_1 \cup C_2$ for all $s > q_3$, then $\widetilde{m_i}(C_1 \cup C_2) = 0$ for all i. Therefore in this case, for each $s > q_3$ and $j \in \mathbb{N}$, $\widetilde{m_i}(A_s) = \widetilde{m_i}(A_s \setminus (C_1 \cup C_2)) \leq \widetilde{m_i}(H_3)$. If not, then there is $l_3 > s_2$ with $A_{l_3} \subset H_3$, and so $\widetilde{m_i}(A_{l_3}) \leq \widetilde{m_i}(H_3)$ for all i. In any case, $\widetilde{m_i}(A_{s_3}) \leq \widetilde{m_i}(H_3)$ for at least an integer $s_3 > s_2$ and for all i. Arguing by induction, we get that to every $k \geq 2$ there corresponds an element A_{s_k} with $\widetilde{m_i}(A_{s_k}) \leq \widetilde{m_i}(H_k)$ for any i, and $s_k < s_{k+1}$ for all $k \geq 2$. This proves the claim.

Thus we get:

$$0 \leq \inf_k \widetilde{m_i}(A_k) \leq \inf_k \widetilde{m_i}(A_{s_k}) \leq \inf_k \widetilde{m_i}(H_k) = 0$$

for each i. Hence the maps $\widetilde{m_i}$, $\omega \notin N$, $i \geq 0$, are σ-additive on $\mathcal{P}(\mathbb{N})$. By the classical Schur lemma, the mappings $\widetilde{m_i}$, $i \in A$, are uniformly σ-additive on $\mathcal{P}(\mathbb{N})$. Hence, $\lim_j [\sup_{i \in A} m_i(H_j)] = \inf_j [\sup_{i \in A} m_i(H_j)] = 0$. $\square$

We now prove the following improvement of the previous result:

Proposition 2.11 *Let $(m_i)_{i \geq 0}$ be a sequence of real-valued σ-additive bounded measures, $\mathcal{I}$ be any admissible ideal, and suppose that there is a set $A' \subset \mathbb{N}$, $A' \in \mathcal{F}(\mathcal{I})$, such that*

$$\text{(A)} \qquad \inf_j [\sup_{i \in A'} v(m_i)(\{j, j+1, \ldots\})] = \lim_j [\sup_{i \in A'} v(m_i)(\{j, j+1, \ldots\})] = 0.$$

If furthermore $\mathcal{I}-\lim_i m_i(H)=m_0(H)$ for every $H\subset\mathbb{N}$, then there is a set $A\in\mathcal{F}(\mathcal{I})$ with the property that $\lim_i m_{i\in A}(H)=m_0(H)$ *uniformly with respect to $H\subset\mathbb{N}$, and moreover*

$$\lim_{i\in A}\left[\sup_q\left(\sum_{j=1}^{q}|m_i(\{j\})-m_0(\{j\})|\right)\right]=0.$$

Proof: First of all observe that, thanks to Proposition 2.8, from pointwise $\mathcal{I}$-convergence of $(m_i)_i$ to m_0 we obtain the existence of a set A'' belonging to $\mathcal{F}(\mathcal{I})$ (the dual filter of $\mathcal{I}$) such that to every $\varepsilon>0$ and $h\in\mathbb{N}$ there corresponds $i_0\in A''$ such that

$$\sum_{q\le h}|m_i(\{q\})-m_0(\{q\})|\le\sum_{q\le h}\frac{\varepsilon}{2^q}<\sum_{q=1}^{\infty}\frac{\varepsilon}{2^q}=\varepsilon \tag{1}$$

whenever $i\ge i_0$, $i\in A''$.

Choose now arbitrarily $\varepsilon>0$. By virtue of property (A) and σ-additivity of m_0 there is $N^*\in\mathbb{N}$ with

$$v(m_i)(\{N+1,N+2,\ldots\})\le\frac{\varepsilon}{4} \tag{2}$$

for any $N\ge N^*$ and $i\in A'\cup\{0\}$. Let now $A:=A'\cap A''$: since $\mathcal{F}(\mathcal{I})$ is a filter, we get that $A\in\mathcal{F}(\mathcal{I})$. Moreover, let $i_0=i_0(\frac{\varepsilon}{2},N^*)$ be the integer associated with $\frac{\varepsilon}{2}$ and N^*, whose existence is guaranteed by (1). Thus we have, for each $i\ge i_0$, $i\in A$ and $H\subset\mathbb{N}$:

$$\begin{aligned}0 &\le |m_i(H)-m_0(H)|\le|m_i(H\cap\{1,...,N^*\})-m_0(H\cap\{1,...,N^*\})|+ \\ &+ v(m_n)(\{N^*+1,N^*+2,...\})+v(m_0)(\{N^*+1,N^*+2,...\})\le\frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.\end{aligned} \tag{3}$$

From (3) we get uniform convergence of $(m_i)_{i\in A}$ to m_0 with respect to $H\subset\mathbb{N}$, that is the first part of the assertion.

In order to prove the last part, let $\nu:\mathcal{P}(\mathbb{N})\to\mathbb{R}$ be the counting measure, and for every $i\in\mathbb{N}\cup\{0\}$, $j\in\mathbb{N}$ set: $f_i(j)=m_i(\{j\})$. Then for each $H\subset\mathbb{N}$ we get

$$\int_H f_i\,d\nu=\sum_{j\in H}f_i(j)=m_i(H).$$

Observe now that uniform convergence of the sequence $(m_i)_{i\in A}$ to m_0 is equivalent to convergence of the involved integrals uniformly with respect to the parameter H which varies in $\mathcal{P}(\mathbb{N})$. Proceeding analogously as in [3, Proposition 3.9], we get convergence in L^1 of $(f_i)_{i\in A}$ to f_0, and therefore we obtain the last part of the assertion. □

3 The basic matrix theorem

We begin with the following lemma, which deals with exchange of limits with respect to $\mathcal{I}$-convergence and holds without assuming necessarily that the involved ideal is a P-ideal (for the classical version, see [8, Lemma I.7.6]).

Lemma 3.1 *Let $(x_{i,j})_{i,j}$ be a double sequence of real numbers, $\mathcal{I}$ be any admissible ideal in $\mathbb{N}$, $\mathcal{F}$ be its dual filter and K be any fixed element of $\mathcal{F}$. Set $\mathcal{I} \times \mathcal{I} := \{D_1 \times D_2 : D_1, D_2 \in \mathcal{I}\}$. Suppose that*

(i) *$\mathcal{I} - \lim_i x_{i,j} = y_j$ exists in $\mathbb{R}$ for all $j \in \mathbb{N}$.*

(ii) *$\mathcal{I} - \lim_j [\sup_{i \in K} |x_{i,j} - x_i|] = 0$.*

Then the following results hold true.

(iii) *There exists in $\mathbb{R}$ the limit $a := \mathcal{I} - \lim_j y_j$.*

(iv) *There exists in $\mathbb{R}$ $b := \mathcal{I} - \lim_i x_i$.*

(v) *There exist an ideal $\mathcal{J} \subset \mathcal{P}(\mathbb{N} \times \mathbb{N})$ and $c \in \mathbb{R}$ such that $\mathcal{I} \times \mathcal{I} \subset \mathcal{J}$ and $\mathcal{J} - \lim_{i,j} x_{i,j} = c$.*

(vi) *There exists in $\mathbb{R}$ $d := \mathcal{I} - \lim_i x_{i,i}$.*

(vii) *We get: $a = b = c = d$.*

Proof: First of all note that, by arguing analogously as in the proof of Proposition 2.8, by (i), to every $\varepsilon > 0$ and $j \in \mathbb{N}$ there corresponds $D_j \in \mathcal{I}$ with $|x_{i,j} - y_j| \le \varepsilon$ whenever $i \notin D_j$.

Moreover, by (ii), for every $\varepsilon > 0$ there is $D \in \mathcal{I}$ such that

$$|x_{i,j} - x_i| \le \varepsilon \quad \text{for all } j \notin D \text{ and } i \in K. \tag{4}$$

We now prove (v). Let $j_0 := \min(\mathbb{N} \setminus D)$. Then by (4) we have:

$$|x_{i,j_0} - x_i| \le \varepsilon \quad \text{for all } i \in K. \tag{5}$$

By (4) and (5) we get that

$$|x_{i,j} - x_{i,j_0}| \le 2\,\varepsilon \quad \text{for all } j \notin D \text{ and } i \in K. \tag{6}$$

By (i) we have the existence in $\mathbb{R}$ of the limit $\mathcal{I} - \lim_i x_{i,j_0} = y_{j_0}$ and so there is $D_{j_0} \in \mathcal{I}$ such that

$$|x_{i,j_0} - y_{j_0}| \le \varepsilon \quad \text{for all } i \notin D_{j_0},\, i \in K. \tag{7}$$

Let $i_0 := \min(\mathbb{N} \setminus D_{j_0})$. Then by (7) we get:

$$|x_{i_0,j_0} - y_{j_0}| \le \varepsilon. \tag{8}$$

By (7) and (8) we obtain:

$$|x_{i,j_0} - x_{i_0,j_0}| \le 2\,\varepsilon \quad \text{for all } i \notin D_{j_0},\, i \in K. \tag{9}$$

By (6) and (9) we get that

$$|x_{i,j} - x_{i_0,j_0}| \le 4\,\varepsilon \quad \text{for all } i \notin D_{j_0}, i \in K \text{ and } j \notin D. \tag{10}$$

Let now $i' \notin D_{j_0}$, $i' \in K$, $j' \notin D$. Then by (10) we have:

$$|x_{i_0,j_0} - x_{i',j'}| \leq 4\,\varepsilon. \tag{11}$$

Let $S := (D_{j_0} \cup (\mathbb{N} \setminus K)) \times D \in \mathcal{I} \times \mathcal{I}$ and

$$\mathcal{J} := \left\{ \bigcup_{s=1}^{k} (A_s \times B_s) : A_s, B_s \in \mathcal{I} \text{ for all } s = 1, \ldots, k;\ k \in \mathbb{N} \right\}.$$

Then $\mathcal{J}$ is an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and $S \in \mathcal{J}$. By (11) we obtain that

$$|x_{i,j} - x_{i',j'}| \leq 8\,\varepsilon \quad \text{for all } (i,j),\ (i',j') \notin S, \tag{12}$$

and by (12) the double sequence $(x_{i,j})_{i,j}$ is $\mathcal{J}$-Cauchy. By Remark 2.3(a), the limit $c := \mathcal{J} - \lim_{i,j} x_{i,j}$ exists in $\mathbb{R}$. Thus (v) is proved.

(vi) With the same notations as in the proof of (v), if i, $i' \notin D_{j_0} \cup D \cup K \in \mathcal{I}$, then from (10) and (11) it follows that

$$|x_{i,i} - x_{i',i'}| \leq 8\varepsilon.$$

Thus the sequence $(x_{i,i})_i$ is $\mathcal{I}$-Cauchy, and hence the limit $\mathcal{I} - \lim_i x_{i,i}$ exists in $\mathbb{R}$ and is equal to c.

We now prove (iii). By (v), for any $\varepsilon > 0$ there is $S \in \mathcal{J}$ with the property that

$$|x_{i,j} - c| \leq \varepsilon \quad \text{for all } (i,j) \notin S. \tag{13}$$

But $S = \bigcup_{s=1}^{k_0} (A_s \times B_s)$, where $k_0 \in \mathbb{N}$ and A_s, $B_s \in \mathcal{I}$ for all $s = 1, \ldots, k_0$. Moreover, by (i), for every $j \in \mathbb{N}$ we have the existence of $D_j \in \mathcal{I}$ with

$$|x_{i,j} - y_j| \leq \varepsilon \quad \text{for all } i \notin D_j. \tag{14}$$

So, for each $i \notin \left(\bigcup_{s=1}^{k_0} A_s\right) \cup D_j \in \mathcal{I}$ and $j \notin \left(\bigcup_{s=1}^{k_0} B_s\right) \in \mathcal{I}$, by (13) and (14) we get:

$$|y_j - c| \leq |x_{i,j} - y_j| + |x_{i,j} - c| \leq 2\,\varepsilon. \tag{15}$$

By (15) we obtain that the element a as in (iii) exists in $\mathbb{R}$ and $a = c$.

(iv) Similarly as in (iii).

(vii) It is an easy consequence of (iii), (iv), (v) and (vi). $\square$

We now turn to the basic matrix theorem.

Theorem 3.2 *Let $(x_{i,j})_{i,j}$ be a bounded double sequence in $\mathbb{R}$, and $\mathcal{I}$ be a P-ideal of $\mathbb{N}$. Assume that:*

(i) $\mathcal{I} - \lim_i x_{i,j} =: x_j$ *exists in $\mathbb{R}$ for all $j \in \mathbb{N}$;*

(ii) $\mathcal{I}-\lim_j x_{i,j}=0$ *for all* $i\in\mathbb{N}$;

(iii) *for every infinite subset* $B\subset\mathbb{N}$ *there is an infinite subset* $C\subset B$ *such that the sequence* $\left(\mathcal{I}-\sum_{j\in C} x_{i,j}\right)_i$ *is convergent.*

Then the following hold:

(I) *There exists* $K\in\mathcal{F}=\mathcal{F}(\mathcal{I})$ *such that* $\mathcal{I}-\lim_i[\sup_{j\in K}|x_{i,j}-x_j|]=0$.

(II) $\mathcal{I}-\lim_j x_j=0$.

(III) *If* $\mathcal{J}\subset\mathcal{P}(\mathbb{N}^2)$ *is the ideal of* $\mathbb{N}^2$ *generated by the finite unions of the Cartesian products of the elements of* $\mathcal{I}$, *then* $\mathcal{J}-\lim_{i,j}x_{i,j}=0$.

(IV) $\mathcal{I}-\lim_i x_{i,i}=0$.

(V) *There is* $A\in\mathcal{F}=\mathcal{F}(\mathcal{I})$ *with* $\mathcal{I}-\lim_j[\sup_{i\in A}|x_{i,j}|]=0$.

Proof: (I) First of all note that, by virtue of (ii) and Proposition 2.8, a set $K\in\mathcal{F}$ can be found, with

$$\lim_{j\in K} x_{i,j}=0 \tag{16}$$

for all $i\in\mathbb{N}$.
From (16) it follows that for any $\varepsilon>0$ and $i,k\in\mathbb{N}$ there is $s=s(i,k)\in K$ with the property that

$$|x_{i,j}-x_{k,j}|\le 2\,\varepsilon \quad \text{for all } j\ge s, j\in K. \tag{17}$$

Moreover, by (i) and Proposition 2.8 again, there is $A\in\mathcal{F}$ with $\lim_{i\in A}x_{i,j}=x_j$ for all $j\in\mathbb{N}$.

Let $A=\{q_1<\ldots<q_i<\ldots\}$: for the sake of simplicity, put $q_i=i$ for all i. Proceeding analogously as above, we have that for every $\varepsilon>0$ and $s\in\mathbb{N}$ there is $p\in\mathbb{N}$ with

$$\sum_{j\in K,j=1}^{s}|x_{i,j}-x_j|\le\sum_{j=1}^{s}\frac{\varepsilon}{2^j}\le\sum_{j=1}^{\infty}\frac{\varepsilon}{2^j}=\varepsilon \quad \text{for all } i\ge p. \tag{18}$$

For all $\varepsilon>0$ and $s\in\mathbb{N}$ there is $p=p(s)\in\mathbb{N}$ with

$$\sum_{j\in K,j=1}^{s}|x_{i,j}-x_{h,j}|\le 2\,\varepsilon \quad \text{for all } i,h\ge p. \tag{19}$$

We will prove that

$$\lim_{i\in A}[\sup_{j\in K}|x_{i,j}-x_j|]=0, \tag{20}$$

where K is as in (16). This, thanks to Proposition 2.7, is enough to prove (I).

Before proving (20), we claim that for every $\varepsilon > 0$ there exists $i \in A$ such that the set

$$\{k \in A : \sup_{j \in K} |x_{i,j} - x_{k,j}| > 4\,\varepsilon\} \tag{21}$$

is finite. Otherwise, there is $\varepsilon > 0$ such that for every $i \in A$ there exist $k = k(i) \in A$, $k > i$ and $j \in K$ with

$$|x_{i,j} - x_{k,j}| > 4\,\varepsilon. \tag{22}$$

Choose arbitrarily $i_1 \in A$: in correspondence with i_1 there exist $k_1 = k(i_1) \in A$, $k_1 > i_1$ and $j_1 \in K$ with

$$|x_{i_1,j_1} - x_{k_1,j_1}| > 4\,\varepsilon. \tag{23}$$

Let $s_1 := s(i_1, k_1) \in K$ be as in (17): without loss of generality, we can choose $s_1 > j_1$. We get

$$|x_{i_1,j} - x_{k_1,j}| \le \varepsilon/2^1$$

whenever $j \ge s_1$, $j \in K$. Let $p_1 := p(s_1)$ be as in (19). We obtain

$$\sum_{j \in K, j=1}^{s_1} |x_{p,j} - x_{q,j}| \le 2\,\varepsilon \quad \text{for all } p, q \ge p_1. \tag{24}$$

Let now $i_2 \in A$, with $i_2 > p_1$. Without loss of generality, we can choose $i_2 \in \{k(i) : i \in \mathbb{N}\}$. In correspondence with i_2 there are $k_2 = k(i_2) \in A$, $k_2 > i_2$, and $j_2 \in K$ such that

$$|x_{i_2,j_2} - x_{k_2,j_2}| > 4\,\varepsilon. \tag{25}$$

Note that, by construction, $j_2 > s_1$. Let $s_2 := s(i_2, k_2) \in K$ be as in (17): without loss of generality, we can choose $s_2 > j_2$. We get

$$\max\{|x_{i_1,j} - x_{k_1,j}|, |x_{i_2,j} - x_{k_2,j}|\} \le \varepsilon/2^2$$

whenever $j \ge s_2$, $j \in K$.

Proceeding by induction, we get the existence of four strictly increasing sequences: $(i_r)_r$ and $(k_r)_r$ in A; $(j_r)_r$ and $(s_r)_r$ in K, with the properties that $i_r < k_r < i_{r+1}$, $j_r < s_r < j_{r+1}$ for all $r \in \mathbb{N}$; $i_r \in \{k(i) : i \in \mathbb{N}\}$ for any $r \ge 2$, and:

(a) $\displaystyle\sum_{j \in K, j=1}^{s_{r-1}} |x_{i_r,j} - x_{k_r,j}| \le 2\,\varepsilon$;

(b) $|x_{i_r,j_r} - x_{k_r,j_r}| > 4\,\varepsilon$;

(c) $|x_{i_r,j_{r+h}} - x_{k_r,j_{r+h}}| \le \displaystyle\sum_{h=1}^{\infty} \varepsilon/2^h = \varepsilon$ for all $r \ge 2$ and $h \in \mathbb{N}$.

By virtue of (iii), in correspondence with ε and $B := \{j_r : r \ge 2\}$ there exist $C \subset B$ and $n_0 \in \mathbb{N}$ such that

$$\left| \mathcal{I} - \sum_{j \in C} (x_{i_r,j} - x_{k_r,j}) \right| \le \varepsilon \tag{26}$$

for all $r \geq n_0$.

Let $u := 2\sup_{i,j}|x_{i,j}|$. By (a), (b) and (c), we get:

$$\begin{aligned}\sum_{j\in C}|x_{i_r,j}-x_{k_r,j}| &\leq \sum_{j\in C, j\in\{j_1,\dots,j_{r-1}\}}|x_{i_r,j}-x_{k_r,j}|+\sum_{j\in C, j\in\{j_{r+1},\dots\}}|x_{i_r,j}-x_{k_r,j}|+|x_{i_r,j_r}-x_{k_r,j_r}| \\ &\leq 2\varepsilon+\varepsilon+u=3\varepsilon+u \qquad (27)\end{aligned}$$

for $r \geq 2$. Therefore, $\sum_{j\in C}|x_{i_r,j}-x_{k_r,j}| \in \mathbb{R}$ and a fortiori $\sum_{j\in C}(x_{i_r,j}-x_{k_r,j}) \in \mathbb{R}$ for such r's.

Moreover, by (iii), $\mathcal{I}-\sum_{j\in C}(x_{i_r,j}-x_{k_r,j}) \in \mathbb{R}$, and so

$$\mathcal{I}-\sum_{j\in C}(x_{i_r,j}-x_{k_r,j})=\sum_{j\in C}(x_{i_r,j}-x_{k_r,j}) \in \mathbb{R}, \quad r \geq 2. \qquad (28)$$

From (26), (27) and (28), if $r \geq n_0$ and $j_r \in C$ then we have:

$$\begin{aligned}|x_{i_r,j_r}-x_{k_r,j_r}| &\leq \left|\sum_{j\in C}(x_{i_r,j}-x_{k_r,j})\right|+\sum_{j\in C, j\in\{j_1,\dots,j_{r-1}\}}|x_{i_r,j}-x_{k_r,j}|+\sum_{j\in C, j\in\{j_{r+1},\dots\}}|x_{i_r,j}-x_{k_r,j}| \\ &= \left|\mathcal{I}-\sum_{j\in C}(x_{i_r,j}-x_{k_r,j})\right|+\sum_{j\in C, j\in\{j_1,\dots,j_{r-1}\}}|x_{i_r,j}-x_{k_r,j}|+\sum_{j\in C, j\in\{j_{r+1},\dots\}}|x_{i_r,j}-x_{k_r,j}| \\ &\leq \varepsilon+2\varepsilon+\varepsilon=4\varepsilon. \qquad (29)\end{aligned}$$

So (29) holds for infinitely many indexes r. This contradicts (22) and proves the claim (21).

We now prove (20). From (21) it follows that the family $\{(x_{i,j})_{i\in A} : j \in K\}$ is $\mathcal{I}_{\text{fin}}$-Cauchy uniformly with respect to $j \in K$. So, the family $\{(x_{i,j})_{i\in A} : j \in K\}$ converges uniformly with respect to $j \in K$, and thus we get (20). This ends the proof of (I).

(II) We have just proved that $\mathcal{I}-\lim_i[\sup_{j\in K}|x_{i,j}-x_j|]=0$, and by (ii) we know that $\mathcal{I}-\lim_j x_{i,j}=0$ for every $i \in \mathbb{N}$. Thus by (iii), (iv), (vi) and (vii) of Lemma 3.1, interchanging the role of the variables i and j, we get that $\mathcal{I}-\lim_j x_j=0$, that is (II).

(III) It is an immediate consequence of (I), (II) and Lemma 3.1.

(IV) It follows from (I) and (vi) of Lemma 3.1.

(V) In the proof of (I) we proved the existence of two elements $A, K \in \mathcal{F}$ such that

$$\lim_{i\in A}[\sup_{j\in K}|x_{i,j}-x_j|]=0. \qquad (30)$$

Moreover, by (II), $\mathcal{I}-\lim_j x_j=0$. Since $\mathcal{I}$ is a P-ideal, by Proposition 2.7 we get: $\mathcal{I}^*-\lim_j x_j=0$, that is a set $K_0 \in \mathcal{F}(\mathcal{I})$ can be found, with $\lim_{j\in K_0} x_j=0$. Let $K' := K \cap K_0$: then $K' \in \mathcal{F}(\mathcal{I})$. In order to prove the assertion, thanks to the first part of Proposition 2.7 it is enough to show that

$$\lim_{j\in K'}[\sup_{i\in A}|x_{i,j}|]=0. \qquad (31)$$

To this aim observe that by (30), for $\varepsilon > 0$ there exists $\bar{i} \in A$ with $|x_{i,j} - x_j| \leq \varepsilon$ whenever $i \in A$, $i \geq \bar{i}$ and $j \in K$ (and a fortiori $j \in K'$). Since $\lim_{j \in K_0} x_j = 0$, to every $\varepsilon > 0$ there corresponds $\bar{j} \in K_0$ such that $|x_j| \leq \varepsilon$ for all $j \geq \bar{j}$, $j \in K_0$ (and a fortiori $j \in K'$). Note that, without loss of generality, the integer $\bar{j}$ can be taken in K'.

Since (ii) holds, proceeding analogously as in the proof of (I) we get: $\lim_{j \in K'} x_{i,j} = 0$ for all $i \in \mathbb{N}$. So, for every $\varepsilon > 0$ and $i = 1, \ldots, \bar{i} - 1$, $i \in A$, there is $j_i \in K'$ with $|x_{i,j}| \leq \varepsilon$ whenever $j \geq j_i$, $j \in K'$.

Let now $j^* := \max\{\bar{j}, \max_{i=1,\ldots,\bar{i}-1, i \in A} j_i\}$, and choose arbitrarily $i \in A$, $j \in K'$, $j \geq j^*$. If $i \geq \bar{i}$, then $|x_{i,j}| \leq |x_{i,j} - x_j| + |x_j| \leq 2\,\varepsilon$. If $i \leq \bar{i} - 1$, then $|x_{i,j}| \leq \varepsilon$. This proves (31) and hence (V), and concludes the proof of the theorem. $\square$

Remark 3.3 Theorem 3.2 extends to the context of P-ideals [1, Theorem 4], which was formulated for $\mathcal{I} = \mathcal{I}_d$.

Moreover note that, if in the hypotheses of Theorem 3.2 we keep (i) and (iii), *fix* $K \in \mathcal{F}$ and replace (ii) with the condition

$$\lim_{j \in K} x_{i,j} = 0 \quad \text{for all } i \in \mathbb{N}, \tag{32}$$

then the thesis of the theorem continues to hold, and the set K for which (I) is fulfilled is just the element K of $\mathcal{F}$ fixed *a priori* in (32): indeed, it will be enough to repeat the same arguments of the proof of 3.2. In particular, if we take $K = \mathbb{N}$, (ii) becomes

(ii') $\lim_j x_{i,j} = 0$ for all $i \in \mathbb{N}$.

Note that, by arguing analogously as in the proof of 3.2 it is possible to prove that (i), (ii') and (iii) imply that

(I') $\mathcal{I} - \lim_i [\sup_{j \in \mathbb{N}} |x_{i,j} - x_j|] = 0$.

Similarly, if in 3.2 we keep (ii) and (iii), fix $A \in \mathcal{F}$ and replace (i) with

$$\lim_{i \in A} x_{i,j} = x_j \quad \text{for all } j \in \mathbb{N},$$

then the set A for which (V) holds is just the mentioned element A of $\mathcal{F}$. In particular, if we choose $A = \mathbb{N}$, (i) becomes

(i') $\lim_i x_{i,j} = x_j$ exists in $\mathbb{R}$ for all $j \in \mathbb{N}$.

Note that, by proceeding analogously as in the proof of 3.2, we can prove that (i'), (ii) and (iii) imply:

(V') $\mathcal{I} - \lim_j [\sup_{i \in \mathbb{N}} |x_{i,j}|] = 0$.

Remark 3.4 We now claim that Theorem 3.2 holds (with $K = \mathbb{N}$) even if we assume (i), (ii') and replace condition (iii) with the following hypothesis:

(iii') *for every strictly increasing sequence* $(n_h)_h$ *in* $\mathbb{N}$ *the sequence*

$$\left(\mathcal{I}-\sum_{j=1}^{\infty} x_{n_h,j}\right)_h$$

$\mathcal{I}$-converges.

We now sketch only the proof of (I), since the proof of the other parts is similar as above.

Let us define the sequence $(k_i)_i$ by setting $k_i = k(i)$, $i \in \mathbb{N}$, where $k(i)$ is as in (22). By (iii') and Proposition 2.6 there is a subsequence $(k_{i_s})_s$ of $(k_i)_i$ such that the sequence

$$\left(\mathcal{I}-\sum_{j=1}^{\infty} x_{k_{i_s},j}\right)_s$$

converges in the ordinary sense. In the argument leading to a contradiction, we take the natural numbers i_r, k_r, in such a way that $i_r \in \{k_{i_s} : s \in \mathbb{N}\}$ for each $r \geq 2$ and $k_r = k(i_r)$ for any $r \in \mathbb{N}$. Note that, proceeding similarly as in (28), it is possible to prove that

$$\mathcal{I}-\sum_{j=1}^{\infty}(x_{i_r,j}-x_{k_r,j}) = \sum_{j=1}^{\infty}(x_{i_r,j}-x_{k_r,j}) \in \mathbb{R}, \quad r \geq 2. \tag{33}$$

From (iii'), the particular choice of the i_r's, k_r's and (33), there is $n_0 \in \mathbb{N}$, such that

$$\left|\sum_{j=1}^{\infty}(x_{i_r,j}-x_{k_r,j})\right| \leq \varepsilon$$

for all $r \geq n_0$. So we obtain:

$$\begin{aligned} |x_{i_r,j_r}-x_{k_r,j_r}| &\leq \left|\sum_{j=1}^{\infty}(x_{i_r,j}-x_{k_r,j})\right| + \sum_{j\in\{j_1,\dots,j_{r-1}\}} |x_{i_r,j}-x_{k_r,j}| + \sum_{j\in\{j_{r+1},\dots\}} |x_{i_r,j}-x_{k_r,j}| \\ &\leq \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon, \end{aligned}$$

getting a contradiction with (22) and proving the claim.

It is not difficult to find an example of bounded real-valued double sequence $(x_{i,j})_{i,j}$, such that for every $B \subset \mathbb{N}$ and for each strictly increasing sequence $(n_h)_h$ the sequence $\left(\sum_{j\in B} x_{n_h,j}\right)_h$ is bounded. If $\mathcal{I}$ is maximal, then such sequences admit always $\mathcal{I}$-limit (see also [4]).

Remark 3.5 Note that, if we assume the continuum hypothesis, then we get a large class of maximal P-ideals (see [10]).

Remark 3.6 Observe that, if the involved ideal $\mathcal{I}$ is not maximal, then the existence of the classical limit l of any bounded real-valued sequence is equivalent to the existence of the $\mathcal{I}$-limits of all its subsequences, and they coincide all with l. Indeed, the following result holds:

Proposition 3.7 *Let $(a_n)_n$ be any bounded sequence in $\mathbb{R}$, $\mathcal{I}$ be any admissible not maximal ideal of $\mathbb{N}$, and suppose that $\mathcal{I}-\lim_h a_{i_h}$ exists in $\mathbb{R}$ for each strictly increasing sequence $(i_h)_h$. Then the ordinary $\lim_n a_n$ exists in $\mathbb{R}$.*

Proof: First of all we claim that, if $\mathcal{I}$ is any not maximal ideal of any infinite set X, then there exist two disjoint elements $B_1 \notin \mathcal{I}$, $B_2 \notin \mathcal{I}$ whose union is X. Otherwise, for each partition of X formed by two elements (B_1, B_2) of $\mathcal{I}$, then either B_1 or B_2 belongs to $\mathcal{I}$. If $B_1 \in \mathcal{I}$, then $B_2 \notin \mathcal{I}$, otherwise X should belong to $\mathcal{I}$, and hence $\mathcal{I}$ should be trivial. Similarly, if $B_2 \in \mathcal{I}$, then $B_1 \notin \mathcal{I}$. Thus the dual filter $\mathcal{F}(\mathcal{I})$ associated with $\mathcal{I}$ should be an ultrafilter, and hence $\mathcal{I}$ should be maximal. This leads to a contradiction and proves the claim. Moreover note that, since $\mathcal{I}$ is admissible, then every finite subset of X belongs to $\mathcal{I}$, and hence the two involved sets B_1, B_2 turn out to be infinite. Thus we can represent them in the form

$$B_1 := \{t_1 < t_2 < \ldots < t_j < \ldots\}, \quad B_2 := \{r_1 < r_2 < \ldots < r_j < \ldots\}. \tag{34}$$

Now suppose by contradiction that $\lim_n a_n$ does not exist in $\mathbb{R}$. So, since $(a_n)_n$ is bounded, there are two sequences in $\mathbb{N}$, $(p'_h)_h$, $(q'_h)_h$ such that $\lim_h a_{p'_h} = l_1$, $\lim_h a_{q'_h} = l_2$, where

$$\liminf_n a_n := l_1 < l_2 := \limsup_n a_n.$$

Set $P := \{p'_j : j \in \mathbb{N}\}$, $Q := \{q'_l : l \in \mathbb{N}\}$. Let now $p_i := p'_1$, and choose $q_1 > p_1$, $q_1 \in Q$: such an element does exist, since Q is infinite. Pick now $p_2 \in P$ such that $p_2 > q_1$: such a choice is possible, because P is infinite. Keeping on by induction, it is possible to construct two sequences $(p_h)_h$, $(q_h)_h$, with the properties that: $\lim_j a_{p_j} = l_1$, $\lim_s a_{q_s} = l_2$, and

$$p_1 < q_1 < p_2 < \ldots < q_{h-1} < p_h < q_h < p_{h+1} < \ldots$$

For example, if we have just defined $p_1 < q_1 < \ldots < p_{h-1} < q_{h-1}$, let us choose $p_h \in P$ such that $p_h > q_{h-1}$ and $q_h \in Q$ with $q_h > p_h$: this is possible, since P and Q are infinite.

Let now B_1, B_2 be as in (34). For every $n \in \mathbb{N}$ there exists one natural number j that $n = t_j$, or there is one positive integer s such that $n = r_s$. In the first case put $b_n := a_{p_j}$, and in the second case set $b_n := a_{q_s}$.

The next step is to prove that for all $l \in \mathbb{R}$ there exists $\delta(l) > 0$, such that

$$\{n \in \mathbb{N} : |b_n - l| > \delta\} \notin \mathcal{I}. \tag{35}$$

First of all, let us consider the case $l \neq l_1$. Take $\delta := \dfrac{|l - l_1|}{2}$ and set $\varepsilon := \dfrac{|l-l_1|}{4} = \dfrac{\delta}{2}$. By the definition of limit, we get: $|a_{p_j} - l_1| \leq \dfrac{|l-l_1|}{4}$ in the complement of a finite number of indexes j. So there exists a finite subset $N_1 \subset \mathbb{N}$ such that, if $n \in B_1 \setminus N_1$, then $|b_n - l_1| \leq \dfrac{|l-l_1|}{4}$. This implies that for all $n \in B_1 \setminus N_1$ we get: $|b_n - l| > \dfrac{|l-l_1|}{2}$. Otherwise we should have:

$$|l - l_1| \leq |l - b_n| + |b_n - l_1| \leq \frac{|l-l_1|}{2} + \frac{|l-l_1|}{4} = \frac{3}{4}|l - l_1|.$$

This is possible if and only if $l = l_1$, but this is absurd, because it contradicts our assumption.

Thus the set $\{n \in \mathbb{N} : |b_n - l| > \delta\}$ contains $B_1 \setminus N_1$, and so it does not belong to $\mathcal{I}$, since $B_1 \notin \mathcal{I}$, N_1 is finite and $\mathcal{I}$ is admissible. Thus (35) is proved, at least when $l \neq l_1$.

We now turn to the case $l = l_1$. Take $\delta := \dfrac{l_2 - l_1}{2}$. Note that $\delta > 0$, since $l_1 < l_2$. Analogously as above, we get $|a_{r_s} - l_2| \leq \dfrac{l_2 - l_1}{4}$ in the complement of finitely many indexes s. Thus there is a finite subset $N_2 \subset \mathbb{N}$ such that $|b_n - l_2| \leq \dfrac{l_2 - l_1}{4}$ whenever $n \in B_2 \setminus N_2$. This implies that for all $n \in B_2 \setminus N_2$ we have: $|b_n - l| > \dfrac{l_2 - l_1}{2}$. Otherwise, we get:

$$0 < l_2 - l_1 \leq |l_1 - b_n| + |b_n - l_2| \leq \frac{l_2 - l_1}{2} + \frac{l_2 - l_1}{4} = \frac{3}{4}(l_2 - l_1) < l_2 - l_1,$$

a contradiction. Thus the set $\{n \in \mathbb{N} : |b_n - l| > \delta\}$ contains $B_1 \setminus N_1$, and so it does not belong to $\mathcal{I}$, since $B_2 \notin \mathcal{I}$, N_2 is finite and $\mathcal{I}$ contains all the finite subsets of $\mathbb{N}$. This proves (35) in the case $l = l_1$.

From (35) it follows that the sequence $(b_n)_n$ does not have $\mathcal{I}$-limit. By construction, it follows easily that the sequence $(a_{p_1}, a_{q_1}, a_{p_2}, \ldots, a_{q_{h-1}}, a_{p_h}, a_{q_h}, a_{p_{h+1}}, \ldots,)$ does not have $\mathcal{I}$-limit. Thus the assertion follows. $\square$

Remark 3.8 We ask whether, if $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ is any fixed admissible ideal, m_i, $i \in \mathbb{N}$, are σ-additive positive measures and $\mathcal{I} - \lim_i m_i(E)$ exists for every $E \in \mathcal{P}(\mathbb{N})$, then for every disjoint sequence $(C_j)_j$ in $\mathcal{P}(\mathbb{N})$ one has: $\mathcal{I} - \lim_j [\sup_{i \in \mathbb{N}} m_i(C_j)] = 0$. The answer is in general negative.

Indeed, let for example $H := \{h_1 < \ldots < h_s < h_{s+1} < \ldots\}$ be an infinite set belonging to $\mathcal{I}$ and such that $\mathbb{N} \setminus H$ is infinite. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, then H does exist. For every $i \notin H$ and $E \subset \mathbb{N}$, set $m_i(E) = 0$. For any $s \in \mathbb{N}$ and $E \subset \mathbb{N}$, set $m_{h_s}(E) = 1$ if $s \in E$ and 0 otherwise. Observe that $m_0(E) := \mathcal{I} - \lim_i m_i(E) = 0$ for each $E \subset \mathbb{N}$. Moreover, it is readily seen that the m_i's are σ-additive positive bounded measures. Indeed, given $i \in \mathbb{N}$ and any disjoint sequence $(C_j)_j$ of subsets of $\mathbb{N}$, the entity $m_i(C_j)$ can be different from zero (and in this case is equal to 1) at most for one index j, because for all $s \in \mathbb{N}$ we get that $m_i(\{s\}) \neq 0$ if and only if $i = h_s$.
For every $j \in \mathbb{N}$ set $C_j := \{j\}$: we get $1 \geq \sup_{i \in \mathbb{N}} m_i(C_j) \geq m_{h_j}(C_j) = 1$. This proves the claim. $\square$

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Σχετικά με έναν ολοκληρωτικό τελεστή.

Επαμεινώνδας Α. Διαμαντόπουλος

Περίληψη

Θεωρούμε τον ολοκληρωτικό τελεστή

$$\mathcal{I}(f)(z) = \frac{1}{[S_z]} \int_{S_z} K(\zeta, z) f(\zeta)\, d\zeta,$$

όπου $[S_z] = (x_2 + \lambda_2 z) - (x_1 + \lambda_1 z)$, $|x_1 \pm \lambda_i| \leq 1$, $i = 1, 2$, και

$$K(\zeta, z) = \frac{1}{p(z)\zeta + q(z)},$$

όπου p, q, μερόμορφές συναρτήσεις στο μοναδιαίο δίσκο. Δείχνουμε πως ο τελεστής $\mathcal{I}$ μπορεί να γραφεί σε όρους σταθμισμένων τελεστών σύνθεσης. Χρησιμοποιώντας την έκφραση αυτή εννοποιούμε παλαιότερα αποτελέσματα που αφορούν τη δράση ειδικών περιπτώσεων του τελεστή αυτού σε χώρους αναλυτικών συναρτήσεων τύπου Hardy, Bergman και Dirichlet .

Σχετικά με έναν ολοκληρωτικό τελεστή.

Επαμεινώνδας Α. Διαμαντόπουλος

7 Ιουνίου 2012

Θεωρούμε τον ολοκληρωτικό τελεστή

$$\mathcal{I}(f)(z) = \frac{1}{[S_z]} \int_{S_z} \frac{f(\zeta)}{p(z)\zeta + q(z)}\, d\zeta, \tag{1}$$

όπου p και q είναι μερόμορφες συναρτήσεις στον μοναδιαίο μιγαδικό δίσκο, x_i, $\lambda_i \in \mathbb{R}$, τέτοια ώστε $|x_i \pm \lambda_i| \leq 1$, και $[S_z] = (x_2 + \lambda_2 z) - (x_1 + \lambda_1 z)$, $z \in \mathbb{D}$. Τελεστές αυτού του τύπου είναι ο ολοκληρωτικός τελεστής του Cesàro

$$\mathcal{C}(f)(z) = \frac{1}{z} \int_0^z \frac{f(\zeta)}{1-\zeta}\, d\zeta,$$

και ο ολοκληρωτικός τελεστής του Hilbert

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(\zeta)}{1-\zeta z}\, d\zeta.$$

Στην παρούσα εργασία μελετούμε τον τελεστή $\mathcal{I}$ στους σταθμισμένους χώρους Dirichlet $\mathcal{D}_\alpha$, $0 < a < 2$, οι οποίοι αποτελούνται από τις αναλυτικές συναρτήσεις f για τις οποίες ισχύει

$$\|f\|^2_{\mathcal{D}_\alpha} = |f(0)|^2 + \iint_{\mathbb{D}} |f'(z)|^2 (1-|z|)^\alpha\, dm(z).$$

Η αλυσίδα των χωρών αυτών περιλαμβάνει το χώρο Hardy H^2, για $\alpha = 1$, και τον κλασσικό χώρο Dirichlet $\mathcal{D}$, για $\alpha = 0$.

Θα αποδείξουμε μία ικανή συνθήκη από την οποία θα συνάγεται πως ο τελεστής $\mathcal{I}$ είναι φραγμένος στους χώρους $\mathcal{D}_\alpha$, $0 < a < 2$.

Η απόδειξη του παραπάνω αποτελέσματος γίνεται με χρήση ενός κατάλληλου μετασχηματισμού με τον οποίο ο τελεστής $\mathcal{I}$ γράφεται σε όρους σταθμισμένων τελεστών σύνθεσης. Η ειδικότερη περίπτωση για απλές γραμμικές συναρτήσεις p και q, παρουσιάστηκε παλαιότερα από τον συγγραφέα στην εργασία [Dia2]. Επιπλέον, στους σταθμισμένους χώρους Dirichlet οι τελεστές Cesàro και Hilbert μελετήθηκαν στις εργασίες [Ga], [Li], κάτι που σημαίνει πως η παρούσα μπορεί να θεωρηθεί ως εννοποίηση των εργασιών αυτών. Τέλος, ειδικές περιπτώσεις του τελεστή $\mathcal{I}$ έχουν μελετηθεί στο παρελθόν σε άλλους χώρους αναλυτικών συναρτήσεων ([Sis2], [Sis4], [DS], [Dia1]).

1 Εισαγωγή.

Στη συνέχεια, με το σύμβολο $\mathcal{X}$ θα συμβολίζουμε ένα χώρο Banach αναλυτικών συναρτήσεων στον οποίο, για κάθε $f \in \mathcal{X}$, και κάθε $z \in \mathbb{D}$, υπάρχει μία σταθερά $c = c(\mathcal{X}) < 1$, τέτοια ώστε

$$|f(z)| \leq \frac{1}{(1-|z|)^c}\|f\|_X. \tag{2}$$

Παραδείγματα τέτοιων χώρων είναι ο χώρος Hardy H^p, $p > 1$ ($c = 1/p$, [Du]), ο χώρος Bergman A^p, $p > 2$ ($c = 2/p$, [Vu]) και ο σταθμισμένος χώρος Dirichlet $\mathcal{D}_\alpha$, $0 < a < 2$, ($c = \alpha/2$, [Ga]).

Λήμμα 1.1. *Έστω x_i, $\lambda_i \in \mathbb{R}$, τέτοια ώστε $|x_i \pm \lambda_i| \leq 1$, $i = 1, 2$ και p, q μερόμορφες συναρτήσεις στο μοναδιαίο δίσκο. Αν για κάθε $z \in \mathbb{D}$,*

$$\Re\left[\frac{p(z)(x_2+\lambda_2 z)+q(z)}{p(z)(x_1+\lambda_1 z)+q(z)}\right]^{1/2} > 0, \tag{3}$$

τότε ο τελεστής $\mathcal{I}$ είναι καλά ορισμένος στο χώρο $\mathcal{X}$.

Απόδειξη. Πρώτα υποθέτουμε πως οι συναρτήσεις p και q είναι αναλυτικές στο

$\mathbb{D}$. Έστω $r_z(t) = [S_z]t + (x_1 + \lambda_1 z)$, $0 < t < 1$, $z \in \mathbb{D}$. Για $f \in X$ και $z \in \mathbb{D}$,

$$\begin{aligned}
|\mathcal{I}(f)(z)| &= \left| \frac{1}{[S_z]} \int_{S_z} \frac{f(\zeta)}{p(z)\zeta + q(z)} \, d\zeta \right| \\
&= \left| \int_0^1 \frac{f(r_z(t))}{p(z) r_z(t) + q(z)} \, dt \right| \\
&\leq \int_0^1 \frac{|f(r_z(t))|}{|p(z) r_z(t) + q(z)|} \, dt \\
&\leq \max_{t \in [0,1]} \frac{C}{|p(z) r_z(t) + q(z)|} \int_0^1 \frac{1}{(1 - |r_z(t)|)^c} \, dt \|f\|_X.
\end{aligned}$$

Ένας απλός υπολογισμός δείχνει πως η υπόθεση (3), με ύψωση στο τετράγωνο, πολλαπλασισμό με -1, πρόσθεση του +1 και μία αντιστροφή, ισοδυναμεί με τη συνθήκη πως η συνάρτηση $|p(z) r_z(t) + q(z)|^{-1}$ είναι φραγμένη ως μιγαδική συνάρτηση της μεταβλητής t, για κάθε $z \in \mathbb{D}$, δηλαδή

$$\max_{t \in [0,1]} \frac{1}{|p(z) r_z(t) + q(z)|} < \infty.$$

Επιπλέον, για κάθε $z \in \mathbb{D}$, και $0 < t < 1$,

$$|r_z(t)| \leq \min\{[S_{\pm 1}]t + (x_1 - \lambda_1), [S_{\pm 1}]t + (x_1 + \lambda_1)\},$$

δηλαδή,

$$1 - |r_z(t)| \geq \max\{1 - (x_1 - \lambda_1) - [S_{\pm 1}]t, 1 - (x_1 + \lambda_1) - [S_{\pm 1}]t\},$$

από το οποίο συνάγουμε

$$\int_0^1 \frac{1}{(1 - |r_z(t)|)^c} \, dt \leq \frac{1}{[1 - (x_1 \pm \lambda_1)]^c} \int_0^1 \frac{1}{\left[1 - \frac{S_{\pm 1}}{1 - (x_1 \pm \lambda_1)} t\right]^c} \, dt.$$

Επιπλέον, καθώς $S_{\pm 1} \leq 1 - (x_1 \pm \lambda_1)$, και $c < 1$, το τελευταίο ολοκλήρωμα είναι πεπερασμένο, δηλαδή ο τελεστής $\mathcal{I}$ είναι καλά ορισμένος για κάθε $z \in \mathbb{D}$ και κάθε $f \in \mathcal{X}$. Τέλος, προσέχουμε πως η υπόθεση της αναλυτικότητας των p ανδ q μπορεί να χαλαρώσει, καθώς στην περίπτωση που μία ή και οι δύο από τις συναρτήσεις αυτές είναι μερόμορφές τότε αρκεί ένας πολλαπλασιασμός των δύο μερών του κλάσματος $f(\zeta)/(p(z)\zeta + q(z))$, με κατάλληλο πολυώνυμο για να συνεχίσουν τα παραπάνω επιχειρήματα να ισχύουν. □

2 Ειδικές περιπτώσεις του τελεστή $\mathcal{I}$

2.1 Ο τελεστής $\mathcal{I}$ ως γενίκευση παλαιότερων ειδικών περιπτώσεων.

Εκτός από τους ολοκληρωτικούς τελεστές του Cesàro και του Hilbert, ο τελεστής $\mathcal{I}$ είναι πρωτότυπο για αρκετούς ακόμα ολοκληρωτικούς τελεστές που έχουν μελετηθεί στο παρελθόν, όπως ο τελεστής $\mathcal{A}$, που είναι ο H^2 συζυγής του τελεστή του Cesàro , ή ο τελεστής $\mathcal{H}_0$, ο οποίος παράγεται από τον αποκομμένο πίνακα του Hilbert. Στον πίνακα 1, παρουσιάζονται οι επιλογές των x_i, λ_i, $i = 1, 2$, p και q που αντιστοιχούν σε κάθε μία τέτοια περίπτωση, μαζί με κάποια αντιπροσωπευτικές εργασίες στις οποίες οι τελεστές αυτοί έχουν μελετηθεί.

Πίνακας 1: Ειδικές περιπτώσεις του τελεστή $\mathcal{I}$.

Τελεστής	x_1	x_2	λ_1	λ_2	$p(z)$	$q(z)$	Άρθρα
$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}\,d\zeta$	0	0	0	1	-1	1	[Ga], [Sis4]
$\mathcal{A}(f)(z) = \frac{1}{z-1}\int_1^z f(\zeta)\,d\zeta$	1	0	0	1	0	1	[Sis2], [Sis3]
$\mathcal{J}(f)(z) = \frac{1}{z-1}\int_1^z \frac{f(\zeta)}{-1-\zeta}\,d\zeta$	1	0	0	1	-1	-1	[Sis1]
$\mathcal{H}(f)(z) = \int_0^1 \frac{f(\zeta)}{1-\zeta z}\,d\zeta$	0	1	0	0	$-z$	1	[DS], [Dia1]
$\mathcal{H}_0(f)(z) = \frac{1}{2}\int_{-1}^1 \frac{f(\zeta)}{1-\zeta z}\,d\zeta$	-1	1	0	0	$-z$	1	[Dia2]

2.2 Ο $\mathcal{I}$ ως τελεστής που παράγεται από τη δράση πίνακα.

Οι περισσότεροι από τους τελεστές που αποτέλεσαν το κίνητρο για την εργασία αυτή είναι τελεστές που παράγονται από τη δράση συγκεκριμένων πινάκων στους συντελεστές αναλυτικών συναρτήσεων. Αναμενόμενα μπορούμε να εντοπίσουμε μία μεγαλύτερη οικογένεια τελεστών του είδους αυτού. Συγκεκριμένα,

έστω
$$M_1 = \begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} & \dots \\ c_{1,0} & c_{1,1} & c_{1,2} & \dots \\ c_{2,0} & c_{2,1} & c_{2,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
όπου
$$c_{n,k} = (-1)^n \frac{p_0^n}{q_0^{n+1}} \frac{x_2^{n+k+1} - x_1^{n+k+1}}{(x_2 - x_1)(n+k+1)}, \quad n, k \geq 0,$$
$-1 \leq x_1 < x_2 \leq 1$, p_0, $q_0 \in \mathbb{R}$, $(q_0 \pm p_0 x_2)(q_0 \pm p_0 x_1) > 0$, και
$$M_2 = \begin{pmatrix} d_{0,0} & 0 & 0 & \dots \\ d_{1,0} & d_{1,1} & 0 & \dots \\ d_{2,0} & d_{2,1} & d_{2,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
όπου,
$$d_{n,k} = \begin{cases} 0, & 0 \leq n < k, \\ \frac{(-1)^{n-k}}{q_0} \left(\frac{p_0}{q_0}\right)^{n-k} \frac{\lambda_2^{n+1} - \lambda_1^{n+1}}{(\lambda_2 - \lambda_1)(n+1)}, & n \geq k, \end{cases},$$
p_0, $q_0 \in \mathbb{R}$, $-1 \leq \lambda_1 < \lambda_2 \leq 1$ και $(q_0 \pm p_0 \lambda_1)(q_0 \pm p_0 \lambda_2) > 0$. Για $p_0 = -q_0$, οι πίνακες της οικογένειας M_1 είναι πίνακες Hankel ενώ μπορούν να θεωρηθούν ως γενίκευση του πίνακα του Hilbert, η περίπτωση του οποίου προκύπτει για την επιλογή $p_0 = -1$, $q_0 = 1$, $x_1 = 0$, $x_2 = 1$. Ο αναγνώστης μπορεί να επιβεβαιώσει πως ο αποκομμένος πίνακας του Hilbert εμφανίζεται με την επιλογή $p_0 = -1$, $q_0 = 1$, $x_1 = -1$, $x_2 = 1$. Από την άλλη μεριά, οι πίνακες της οικογένειας M_2 είναι κάτω τριγωνικοί πίνακες οι οποίοι μπορεί να θεωρηθούν ως γενίκευση του πίνακα Cesàro ο οποίος προκύπτει για $p_0 = -1$, $q_0 = 1$, $\lambda_1 = 0$, $\lambda_2 = 1$.

Έστω, τώρα $\mathcal{X}$ χώρος για τον οποίο ισχύει η υπόθεση (2). Για κάθε αναλυτική συνάρτηση $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{X}$, έστω
$$\mathcal{M}_1 : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k c_{n,k} z^n,$$
και
$$\mathcal{M}_2 : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k d_{n,k} z^n.$$

Επιπλέον, υποθέτουμε πως για κάθε $f(z) = \sum_{n\geq} a_n z^n$, τα παραπάνω άπειρα αθροίσματα συγκλίνουν και ορίζουν αναλυτικές συναρτήσεις για κάθε $f \in \mathcal{X}$. Πράγματι, αυτό μπορεί να επαληθευτεί για τους χώρους Hardy , Bergman και τους σταθμισμένους χώρους Dirichlet . Τώρα υπολογίζουμε

$$\begin{aligned}
\mathcal{M}_1(f)(z) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k(-1)^n \frac{p_0^n}{q_0^{n+1}} \frac{x_2^{n+k+1} - x_1^{n+k+1}}{(x_2-x_1)(n+k+1)} z^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} a_k(-1)^n \frac{p_0^n}{q_0^{n+1}} \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \zeta^{n+k}\, d\zeta\right) z^n \\
&= \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \frac{f(\zeta)}{p_0 z\zeta + q_0}\, d\zeta,
\end{aligned}$$

και

$$\begin{aligned}
\mathcal{M}_2(f)(z) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k \frac{(-1)^{n-k}}{q_0}\left(\frac{p_0}{q_0}\right)^{n-k} \frac{\lambda_2^{n+1} - \lambda_1^{n+1}}{(\lambda_2-\lambda_1)(n+1)} z^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k \frac{(-1)^{n-k}}{q_0}\left(\frac{p_0}{q_0}\right)^{n-k} \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1 z}^{\lambda_2 z} \zeta^n\, d\zeta\right) z^n \\
&= \frac{1}{(\lambda_2-\lambda_1)z}\int_{\lambda_1 z}^{\lambda_2 z} \frac{f(\zeta)}{p_0\zeta + q_0}\, d\zeta.
\end{aligned}$$

Από τις υποθέσεις και το Λήμμα 1.1 συνάγουμε πως οι τελεστές $\mathcal{M}_1$, $\mathcal{M}_2$ είναι καλά ορισμένοι στο χώρο $\mathcal{X}$. Τέλος, καθώς από τις παραπάνω συνθήκες, η μορφή των τελεστών $\mathcal{M}_1$, $\mathcal{M}_2$ ως σειρά συγκλίνει, η αλλαγή του ολοκληρώματος με τη σειρά είναι εφικτή και καταλήγουμε στο επιθυμητό αποτέλεσμα πως οι τελεστές αυτοί αναγνωρίζονται ως ειδικές περιπτώσεις του $\mathcal{I}$.

3 Ο τελεστής $\mathcal{I}$ σε όρους σταθμισμένων τελεστών σύνθεσης.

Υπενθυμίζουμε πως ο $\mathcal{X}$ είναι ένας χώρος αναλυτικών συναρτήσεων για τον οποίον η υπόθεση (2) ισχύει. Για κάθε $(t,z) \in (0,1) \times \mathbb{D}$, ορίζουμε

$$w(t,z) = \frac{1}{(x_2 + \lambda_2 z - t[S_z])p(z) + q(z)},$$

και

$$\gamma(t,z)=\frac{(x_1+\lambda_1 z)(x_2+\lambda_2 z)p(z)+(x_1+\lambda_1 z+t[S_z])q(z)}{(x_2+\lambda_2 z-t[S_z])p(z)+q(z)}.$$

Πρόταση 3.1. *Εστω λ_i, $x_i\in[-1,1]$, $i=1,2$, $|x_i\pm\lambda_i|\leq 1$, $i=1,2$, p, q μερόμορφες στο $\mathbb{D}$, τέτοια ώστε*

$$\Re\left[\frac{p(z)(x_2+\lambda_2 z)+q(z)}{p(z)(x_1+\lambda_1 z)+q(z)}\right]^{1/2}>0.$$

και για κάθε $(t,z)\in(0,1)\times\mathbb{D}$,

$$|(x_1+\lambda_1 z)(x_2+\lambda_2 z)p(z)+(x_1+\lambda_1 z+t[S_z])q(z)|<|(x_2+\lambda_2 z-t[S_z])p(z)+q(z)|.$$

Τότε, για κάθε $f\in\mathcal{X}$,

$$\mathcal{I}(f)(z)=\int_0^1 T_t(f)(z)\,dt,$$

όπου

$$T_t(f)(z)=w(t,z)f(\gamma(t,z)).$$

Απόδειξη. Από την πρώτη υπόθεση και το Λήμμα 1.1 συνάγουμε πως ο τελεστής $\mathcal{I}$ είναι καλά ορισμένος στο χώρο $\mathcal{X}$, ενώ από τη δεύτερη υπόθεση, η συνάρτηση γ είναι μία καλά ορισμένη απεικόνιση του μοναδιαίου δίσκου. Εύκολα επαληθεύουμε πως $\gamma(0,z)=x_1+\lambda_1 z$, και $\gamma(1,z)=x_2+\lambda_2 z$. Στο ολοκλήρωμα (1) εφαρμόζουμε την αλλαγή μεταβλητής $\zeta\to\gamma(t,z)$, και υπολογίζουμε,

$$\mathcal{I}(f)(z)=\frac{1}{[S_z]}\int_0^1\frac{f(\gamma(t,z))}{p(z)\gamma(t,z)+q(z)}\frac{\partial\gamma(t,z)}{\partial t}\,dt.$$

Είναι

$$p(z)\gamma(t,z)+q(z)=\frac{[p(z)(x_1+\lambda_1 z)+q(z)][p(z)(x_2+\lambda_2 z)+q(z)]}{p(z)(x_2+\lambda_2 z-t[S_z])+q(z)},$$

και

$$\frac{\partial\gamma(t,z)}{\partial t}=\frac{[S_z][p(z)(x_1+\lambda_1 z)+q(z)][p(z)(x_2+\lambda_2 z)+q(z)]}{[p(z)(x_2+\lambda_2 z-t[S_z])+q(z)]^2}.$$

Ορισμένες απλές πράξεις μας δίνουν

$$
\begin{aligned}
\mathcal{I}(f)(z) &= \int_0^1 \frac{f(\gamma(t,z))}{p(z)(x_2+\lambda_2 z - t[S_z]) + q(z)}\,dt \\
&= \int_0^1 w(t,z) f(\gamma(t,z))\,dt,
\end{aligned}
$$

που είναι το επιθυμητό αποτέλεσμα. □

4 Εκτίμηση της νόρμας του τελεστή $\mathcal{I}$ στους σταθμισμένους χώρους Dirichlet.

Στην παράγραφο αυτή βρίσκουμε την επιθυμητή εκτίμηση. Επικεντρωνόμαστε στην περίπτωση $\mathcal{X} = \mathcal{D}_\alpha$, $0 < \alpha < 2$. Υπενθυμίζουμε την ανισότητα Schwarz's-Pick,

$$\frac{1-|z|}{1-|\gamma(t,z)|} \leq \frac{1}{|\partial_z \gamma(t,z)|}, \quad (t,z) \in (0,1) \times \mathbb{D},$$

η οποία θα χρησιμοποιηθεί παρακάτω.

Λήμμα 4.1. *Έστω $f \in \mathcal{D}_\alpha$, $0 < \alpha < 2$, και για κάθε $(t,z) \in (0,1) \times \mathbb{D}$,*

$$|(x_1+\lambda_1 z)(x_2+\lambda_2 z)p(z) + (x_1+\lambda_1 z + t[S_z])q(z)| < |(x_2+\lambda_2 z - t[S_z])p(z) + q(z)|.$$

Τότε για $0 < \alpha < 2$,

$$\|T_t(f)\|^2_{\mathcal{D}_\alpha} \leq C \left[\int_{\mathbb{D}} \frac{|\partial_z w(t,z)|^2}{|\partial_z \gamma(t,z)|^\alpha}\, dm(z) + \sup_{z\in\mathbb{D}} \frac{|w(t,z)|^2}{|\partial_z \gamma(t,z)|^\alpha} \right] \|f\|^2_{\mathcal{D}_\alpha},$$

όπου C είναι κατάλληλη σταθερά ανεξάρτητη από το t.

Απόδειξη. Για την απόδειξη συμβολίζουμε $w_t(z) = w(t,z)$ και $\gamma_t(z) = \gamma(t,z)$. Έστω $f \in \mathcal{D}_\alpha$, $0 < \alpha < 2$. Είναι

$$
\begin{aligned}
\|T_t(f)\|^2_{\mathcal{D}_\alpha} &= |T_t(f)(0)|^2 + \int_{\mathbb{D}} |(w_t(z) f(\gamma_t(z)))'|^2 (1-|z|)^\alpha\, dm(z) \\
&\leq |T_t(f)(0)|^2 + 2\int_{\mathbb{D}} |w_t(z)'|^2 |f(\gamma(t,z))|^2 (1-|z|)^\alpha\, dm(z) \\
&\quad + 2\int_{\mathbb{D}} |w(t,z)|^2 |(f(\gamma_t(z)))'|^2 (1-|z|)^\alpha\, dm(z) \\
&= |T_t(f)(0)|^2 + 2I_1 + 2I_2.
\end{aligned}
$$

Τώρα,

$$
\begin{aligned}
I_1 &= \int_{\mathbb{D}} |w_t'(z)|^2 |f(\gamma_t(z))|^2 (1-|z|)^\alpha \, dm(z) \\
&\leq C \int_{\mathbb{D}} \frac{(1-|z|)^\alpha |w_t'(z)|^2}{(1-|\gamma_t(z)|)^\alpha} \, dm(z) \|f\|^2_{\mathcal{D}_\alpha} \\
&\leq C \int_{\mathbb{D}} \frac{|w_t'(z)|^2}{|\gamma_t'(z)|^\alpha} \, dm(z) \|f\|^2_{\mathcal{D}_\alpha},
\end{aligned}
$$

ενώ για το ολοκλήρωμα I_2 βρίσκουμε,

$$
\begin{aligned}
I_2 &= \int_{\mathbb{D}} |w_t(z)|^2 |f'(\gamma_t(z))|^2 |\gamma_t'(z)|^2 (1-|z|)^\alpha \, dm(z) \\
&= \int_{\mathbb{D}} |w_t(z)|^2 |f'(\gamma_t(z))|^2 |\gamma_t'(z)|^2 (1-|\gamma_t(z)|)^\alpha \frac{(1-|z|)^\alpha}{(1-|\gamma_t(z)|)^\alpha} \, dm(z) \\
&\leq \sup_{z\in\mathbb{D}} \frac{|w_t(z)|^2}{|\gamma_t'(z)|^\alpha} \int_{\mathbb{D}} |f'(\gamma_t(z))|^2 |\gamma_t'(z)|^2 (1-|\gamma_t(z)|)^\alpha \, dm(z) \\
&\leq \sup_{z\in\mathbb{D}} \frac{|w_t(z)|^2}{|\gamma_t'(z)|^\alpha} \|f\|^2_{\mathcal{D}_\alpha}.
\end{aligned}
$$

Πίνακας 2: Αναγκαίοι υπολογισμοί σχετικά με τους τελεστές που αποτέλεσαν το κίνητρο για την εργασία αυτή.

Τελεστής	$\left[\frac{p(z)(x_2+\lambda_2 z)+q(z)}{p(z)(x_1+\lambda_1 z)+q(z)}\right]^{1/2}$	$\gamma(t,z)$
$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}\, d\zeta$	$\sqrt{1-z}$	$\frac{tz}{(t-1)z+1}$
$\mathcal{A}(f)(z) = \frac{1}{z-1}\int_1^z f(\zeta)\, d\zeta$	1	$tz+1-t$
$\mathcal{J}(f)(z) = \frac{1}{z-1}\int_1^z \frac{f(\zeta)}{-1-\zeta}\, d\zeta$	$\sqrt{\frac{1+z}{2}}$	$\frac{(-t-1)z+t-1}{(t-1)z-t-1}$
$\mathcal{H}(f)(z) = \int_0^1 \frac{f(\zeta)}{1-\zeta z}\, d\zeta$	$\sqrt{1-z}$	$\frac{t}{(t-1)z+1}$
$\mathcal{H}_0(f)(z) = \frac{1}{2}\int_{-1}^1 \frac{f(\zeta)}{1-\zeta z}\, d\zeta$	$\sqrt{\frac{1-z}{1+z}}$	$\frac{2z+4t-2}{(4t-2)z+2}$

Τελικά,

$$|T_t(f)(0)|^2 = |w_t(0)|^2|f(\gamma_t(0))|^2 \leq \frac{C|w_t(0)|^2}{(1-|\gamma_t(0)|)^\alpha}\|f\|^2_{\mathcal{D}_\alpha}$$
$$\leq \frac{C|w_t(0)|^2}{|\gamma_t'(0)|^\alpha}\|f\|^2_{\mathcal{D}_\alpha} \leq C\sup_{z\in\mathbb{D}}\frac{|w_t(z)|^2}{|\gamma_t'(z)|^\alpha}\|f\|^2_{\mathcal{D}_\alpha},$$

από το οποίο παίρνουμε και το επιθυμητό αποτέλεσμα. □

Από το τελευταίο Λήμμα, την Πρόταση 3.1 και την ανισότητα του Minkowski εύκολα αποδεικνύουμε το

Θεώρημα 4.1. *Έστω* λ_i, $x_i \in [-1,1]$, $i = 1,2$, $|x_i \pm \lambda_i| \leq 1$, $i = 1,2$, p, $q \in H(\mathbb{D})$, *τέτοια ώστε*

$$\Re\left[\frac{p(z)(x_2+\lambda_2 z)+q(z)}{p(z)(x_1+\lambda_1 z)+q(z)}\right]^{1/2} > 0,$$

και για κάθε $(t,z) \in (0,1)\times\mathbb{D}$,

$$\frac{|(x_1+\lambda_1 z)(x_2+\lambda_2 z)p(z)+(x_1+\lambda_1 z+t[S_z])q(z)|}{|(x_2+\lambda_2 z - t[S_z])p(z)+q(z)|} < 1. \tag{4}$$

Τότε, για κάθε $f \in \mathcal{D}_\alpha$, $0<\alpha<2$,

$$\|\mathcal{I}(f)\|_{\mathcal{D}_\alpha} \leq C\int_0^1\left[\int_{\mathbb{D}}\frac{|\partial_z w(t,z)|^2}{|\partial_z\gamma(t,z)|^\alpha}dm(z)+\sup_{z\in\mathbb{D}}\frac{|w(t,z)|^2}{|\partial_z\gamma(t,z)|^\alpha}\right]^{1/2}dt\|f\|_{\mathcal{D}_\alpha}.$$

Θα εφαρμόσουμε το παραπάνω θεώρημα για να δείξουμε πως οι τελεστές του Πίνακα 2 είναι φραγμένοι στο σταθμισμένο χώρο Dirichlet.

Πόρισμα 4.1. *Οι τελεστές* $\mathcal{C}$, $\mathcal{A}$, $\mathcal{J}$, $\mathcal{H}$ *και* $\mathcal{H}_0$ *είναι φραγμένοι στο σταθμισμένο χώρο Dirichlet,* $\mathcal{D}_\alpha$, $0<\alpha<2$.

Απόδειξη. Εύκολα επαληθεύουμε πως οι παραπάνω τελεστές είναι καλά ορισμένοι στους χώρους αυτούς (Πίνακας 2). Χρησιμοποιώντας κλασσικές τεχνικές δείχνουμε πως τα επιμέρους αντίστοιχα ολοκληρώματα είναι φραγμένα για κάθε έναν από τους τελεστές. Το αποτέλεσμα προκύπτει από το Θεώρημα 4.1. □

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H^p BOUNDS FOR SPECTRAL MULTIPLIERS

ON RIEMMANIAN MANIFOLDS

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ANALYSIS CONFERENCE. IOANNINA GREECE

- Let $m(\lambda)$ be a bounded measurable function in $\mathbb{R}^n$ and let T_m be the operator defined by

$$\widehat{T_m f}(\lambda) = m(\lambda)\widehat{f}(\lambda).$$

The Mikhlin-Hörmander multiplier theorem (M-H 1960) asserts that if the multiplier $m(\lambda)$ satisfies the condition

$$\sup_{\lambda \in \mathbb{R}^n} |\lambda|^{\alpha} |\partial^{\alpha} m(\lambda)| < \infty,$$

for any multi-index α, with $|\alpha| \leq \left[\frac{n}{2}\right] + 1$, then T_m is bounded on L^p, $1 < p < \infty$ and from L^1 to L^1_w.

Calderón and Torchinsky extending this theorem (C-T 1977) , proved that if the multiplier $m(\lambda)$ satisfies the condition

$$\sup_{\lambda \in \mathbb{R}^n} |\lambda|^{\alpha} |\partial^{\alpha} m(\lambda)| < \infty,$$

for any multi-index α, with $|\alpha| \leq n\left[\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1$, then T_m is bounded on $H^p, 0 < p \leq 1$.

- There are many generalizations of those theorems. For example on Manifolds (M-H), Discrete groups, Lie groups, Nilpotent groups, Symmetric spaces, Graphs, Stratified groups e.a....

- My generalization (2010) of (C-T) is on the context of Riemannian manifolds

- Let M be a $n-$dimensional, complete, noncompact Riemannian manifold with $\mathcal{C}^{\infty}-$bounded geometry.We denote by $d(.,.)$ the Riemannian distance, by dx the Riemannian measure, by $B(x,r)$ the ball centered at $x \in M$ with radius $r > 0$ and by $V(x,r)$ its volume.

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- We assume that M satisfies the **doubling volume property**, i.e. there is a constant $c > 0$, such that

$$V(x, 2r) \leq cV(x, r), \ \forall\, x \in M, \ r > 0. \tag{0.1}$$

From (0.1) it follows that there exist constants $c, D > 0$, such that

$$\frac{V(x,r)}{V(x,t)} \leq c\left(\frac{r}{t}\right)^{D}, \ \forall x \in M, \ r \geq t > 0. \tag{0.2}$$

- Let us denote by Δ the **Laplace-Beltrami operator** on M and by $p_t(x, y)$, $t > 0$, $x, y \in M$, the **heat kernel** of M, i.e. the fundamental solution of the heat equation $\partial_t u = \Delta u$. We assume that $p_t(x, y)$ satisfies the following **estimates**: there are constants $c, c' > 0$ such that

$$p_t(x, y) \leq c' \frac{e^{-d(x,y)^2/ct}}{V(x, \sqrt{t})}, \tag{0.3}$$

for all $t > 0$ and $x, y \in M$, and there are constants $c_1, c_2 > 0$ and $\gamma \in (0, 1)$, such that for all $t > 0$, and $x, y, z \in M$, with $d(y, z) \leq \sqrt{t}$,

$$|p_t(x, y) - p_t(x, z)| \leq \frac{c_1 e^{-c_2 d(x,y)^2/t}}{V(x, \sqrt{t})} \left(\frac{d(y, z)}{\sqrt{t}}\right)^{\gamma}. \tag{0.4}$$

- The Laplace-Beltrami operator Δ on M is a positive and selfadjoint operator on $L^2(M)$. Thus, by the spectral theorem

$$\Delta = \int_0^\infty \lambda dE_\lambda,$$

where dE_λ is the spectral measure on M.

If $m : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, by the spectral theorem we can define the operator

$$m(\Delta) = \int_0^\infty m(\lambda) dE_\lambda,$$

which is a bounded operator on $L^2(M)$, with $||m(\Delta)||_{2\to 2} \leq ||m||_\infty$. The function m is called a multiplier and the operator $m(\Delta)$, is called a spectral multiplier.

- Let us set,

$$p_0 = \frac{D}{D + \gamma},$$

and

$$A = A(p) = D\left(\frac{1}{p} - \frac{1}{2}\right) + \varepsilon, \ \varepsilon > 0,$$

for all $p \in (p_0, 1]$. Note that in case when $\mathrm{Ric}\,(M) \geq 0, p_0 = \frac{n}{n+1}$.

- Let us denote by $C^A(\mathbb{R})$ the Lipschitz space of order $A > 0$, and by $H^p(M)$ the Hardy space. Finally, let us fix a function $0 \leq \phi \in C^\infty(\mathbb{R})$, with
$$\phi(t) = 1, \ \forall t \in [1,2], \quad \phi(t) = 0, \ t \in \left(\tfrac{1}{2}, 4\right)^c.$$
In the present work we prove the following

- **theorem:** Let M be a Riemannian manifold as above and let $m(\lambda)$, $\lambda \in \mathbb{R}$, be a multiplier satisfying

(0.5) $$\sup_{t>0} ||\phi(.)m(t.)||_{C^{A(p)}} < \infty, \quad p \in (p_0, 1]$$

Then the operator $m(\Delta)$ is bounded on H^p.

We note that by interpolation and duality, from Theorem it follows that $m(\Delta)$ is bounded $L^p(M)$, for $1 < p < \infty$, and on $BMO(M)$.

- exg. $\Delta^{i\beta}$, $\beta \in \mathbb{R}$.

- If $p \in (p_0, 1]$, we say that a function a is a p-atom, if there is a ball $B(y,r)$ such that

(0.6) $$supp(a) \subseteq B(y,r), \ ||a||_\infty \leq V(y,r)^{-1/p}$$

and $\int_M a(x)dx = 0$. From (0.6) we get that

(0.7) $$||a||_q \leq V(y,r)^{(1/q)-(1/p)}, \quad q \geq 1.$$

- We need first to define the **Lipschitz space** $\mathcal{L}_\alpha$, $\alpha > 0$. We say that $f \in \mathcal{L}_\alpha$, if there is a constant $c > 0$ such that for every ball B and $x, y \in B$, we have

(0.8) $$|f(x) - f(y)| \leq c|B|^\alpha.$$

The norm $||f||_{\mathcal{L}_\alpha}$ is defined as the smallest of those constants c and makes $\mathcal{L}_\alpha$,a Banach space.

For $p \in (p_0, 1)$ we set $\alpha = (1/p) - 1$. Then we define H^p as the space of those functionals $f \in \mathcal{L}_\alpha'$ which can be written as $f = \sum_{n=0}^{\infty} \lambda_n a_n$, where $(\lambda_n) \in \ell^p$ and (a_n) is a sequence of p-atoms. We set
$$||f||_{H^p} = \inf\left\{ \left(\sum_{n=0}^{\infty} |\lambda_n|^p\right)^{1/p} ; \ f = \sum_{n=0}^{\infty} \lambda_n a_n \right\}.$$

We note that the dual H^p is $\mathcal{L}_\alpha$ and that for every $f \in \mathcal{L}_\alpha$, and for every ball B and $y \in B$, we have that

(0.9) $$||f - f(y)||_{L^2(B)} \leq ||f||_{\mathcal{L}_\alpha} |B|^{(1/p)-(1/2)}.$$

- **Strategy of the proof**
  (1) Let p be in $(p_0, 1)$, a be a p-atom supported on $B(y,r)$, $y \in M$, $r > 0$ and $\psi \in C_0^\infty$. By the duality argument it suffices to show that
$$|\langle m(\Delta)a, \psi\rangle| \leq c||a||_{H^p}||\psi||_{\mathcal{L}_\alpha} = c||\psi||_{\mathcal{L}_\alpha},$$

(2) Cancelation property: For every p-atom a, we have

$$\int_M (m(\Delta)a)(x)dx = 0.$$

Then we write

$$\langle m(\Delta)a, \psi\rangle = \langle m(\Delta)a, \psi - \psi(y)\rangle. \tag{0.10}$$

and $\psi - \psi(y) = \psi_1 + \psi_2$, supported on ball $B(y, 4r)$ and on its complement respectively.

We have then

$$\langle m(\Delta)a, \psi\rangle = \langle m(\Delta)a, \psi_1\rangle + \langle m(\Delta)a, \psi_2\rangle. \tag{0.11}$$

(3) By the Cauchy-Schwarz inequality we get that $|\langle m(\Delta)a, \psi_1\rangle| \leq$

$$||m(\Delta)||_{2\to 2}||a||_2||\psi - \psi(y)||_{L^2(B(y,4r))}.$$

Using (0.7) and (0.9), it follows from the doubling property that

$$|\langle m(\Delta)a, \psi_1\rangle| \leq c||\psi||_{\mathcal{L}_\alpha}.$$

(4) We cut the multiplier on compactly supported terms m_j and

$$|\langle m(\Delta)a, \psi_2\rangle| \leq \sum_{j<N+4} |\langle m_j(\Delta)a, \psi_2\rangle| + \sum_{j\geq N+4} |\langle m_j(\Delta)a, \psi_2\rangle|,$$

where $N \in \mathbb{Z}$ be such that

$$2^{N/2} \leq r < 2^{(N+1)/2}. \tag{0.12}$$

(5) The second sum is estimated simillarly with the case of graphs.

(6) For the first sum because, $B(y, 4r)^c \subseteq \cup_{q\geq N+4} A_q(y)$, where

$$A_q(y) = B(y, 2^{(q+1)/2}) - B(y, 2^{q/2}),$$

we take by the Cauchy- Swartz

$$|\langle m_j(\Delta)a, \psi_2\rangle| \leq$$

$$\sum_{q\geq N+4} ||m_j(\Delta)a||_{L^2(A_q(y))}||\psi_2||_{L^2(A_q(y))},$$

and by Minkowski inequality , $||m_j(\Delta)a||_{L^2(A_q(y))} \leq$

$$||a||_1 \sup_{d(z,y)\leq r} ||K_j(.,z)||_{L^2(A_q(y))}.$$

Where K_j is the kernel of $m_j(\Delta)$.It suffices to estimate the norm

$$||K_j(.,z)||_{L^2(A_q(y))}$$

but this is a consequence of heat kernel's estimates. In fact we have if $j \leq q$,

$$||K_j(.,y)||_{L^2(B(y,2^{q/2})^c)} \leq \frac{c||m_j||_{C^A}2^{-A(q-j)/2}}{\sqrt{V(y, 2^{j/2})}}. \tag{0.13}$$

Putting all together with the relations (0.7),(0.9), using the doubling volume property and summing over q and j we have that

$$|\langle m(\Delta)a, \psi_2\rangle| \leq c||\psi||_{\mathcal{L}_\alpha}(q.e.d.)$$

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ΜΙΑ ΝΕΑ ΚΛΑΣΗ ΑΡΙΘΜΗΣΙΜΑ ΚΑΘΟΡΙΖΟΜΕΝΩΝ ΧΩΡΩΝ BANACH

ΚΑΜΠΟΥΚΟΣ ΚΥΡΙΑΚΟΣ–ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΑΣ

Πρόταση 1. *Έστω X υπόχωρος ενός συμπαγούς τοπολογικού χώρου K. Τότε τα ακόλουθα είναι ισοδύναμα:*

(i) *Υπάρχει ακολουθία K_n, $n \in \mathbb{N}$ συμπαγών υποσυνόλων του K τέτοια ώστε για κάθε συμπαγές υποσύνολο L του X και $x \in K \setminus X$ να υπάρχει $n \in \mathbb{N}$ με $L \subseteq K_n$ και $x \notin K_n$.*

(ii) *Υπάρχουν συμπαγή υποσύνολα B_s, $s \in S$ του K και υποσύνολο Σ' του Σ τέτοια ώστε για κάθε συμπαγές υποσύνολο L του X να υπάρχει $\sigma \in \Sigma'$ με*

$$L \subseteq \bigcap_{n=1}^{\infty} B_{\sigma|n} \subseteq X \quad \textit{και} \quad X = \bigcup_{\sigma\in\Sigma'} \bigcap_{n=1}^{\infty} B_{\sigma|n}$$

(iii) *Υπάρχει υποσύνολο Σ' του Σ και άνω ημισυνεχής συνάρτηση $F : \Sigma' \to \mathcal{K}(X)$ τέτοια ώστε για κάθε συμπαγές υποσύνολο L του X να υπάρχει $\sigma \in \Sigma'$ με $L \subseteq F(\sigma)$. Ειδικότερα $X = F(\Sigma')$.*

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Ορισμός 2. Ένας τ.χ. X καλείται *ισχυρά αριθμήσιμα καθοριζόμενος (SCD)* αν υπάρχει ένας τ.χ. K τέτοιος ώστε $X \subseteq K$ και οι συνθήκες της προηγούμενης πρότασης να ικανοποιούνται.

Παρατήρηση 3. (i) Η παραπάνω πρόταση ισχύει αν αντί για ζεύγη (L, x), με L συμπαγές υποσύνολο του X και $x \in K \setminus X$, στη συνθήκη (i), θεωρήσουμε ζεύγη (u, x) με $u \in X$ και $x \in K \setminus X$ (και ανάλογες μετατροπές στις συνθήκες (ii) και(iii)). Στην περίπτωση αυτή έχουμε την κλασική έννοια του Αριθμήσιμα Καθοριζόμενου χώρου.
(ii) Σημειώνουμε επίσης ότι αν οι συνθήκες (ii) κα (iii) ικανοποιούνται από το Σ αντί του Σ' τότε έχουμε την έννοια του ισχυρά $\mathcal{K}$-αναλυτικού τοπολογικού χώρου που εισάγεται από τους Μερκουράκη – Σταμάτη.

Παραδείγματα 4. (i) Κάθε ισχυρά $\mathcal{K}$–αναλυτικός τοπολογικός χώρος είναι ισχυρά αριθμήσιμα καθοριζόμενος.
(ii) Κάθε διαχωρίσιμος μετρικός χώρος είναι ισχυρά αριθμήσιμα καθοριζόμενος.

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(iii) Ένας $\mathcal{K}$–αναλυτικός χώρος δεν είναι κατ' ανάγκη ισχυρά αριθμήσιμα καθοριζόμενος. Έστω Γ ένα σύνολο με $|\Gamma| = c$ (την ισχύ του συνεχούς) και $X = [\Gamma]^{<\omega}$ (το σύνολο των πεπερασμένων υποσυνόλων του Γ),θεωρούμενο ως υπόχωρος του συμπαγούς χώρου $\{0,1\}^{\Gamma}$. Ο χώρος X είναι σ-συμπαγής (ειδικότερα $\mathcal{K}$–αναλυτικός), διότι μπορεί να γραφεί στη μορφή $X = \bigcup_{n=1}^{\infty}[\Gamma]^{\leq n}$, όπου $[\Gamma]^{\leq n}$ είναι το σύνολο των υποσυνόλων του Γ με $|A| \leq n$, ενώ δεν είναι ισχυρά αριθμήσιμα καθοριζόμενος.

Βασικές ιδιότητες

(i) Κλειστός υπόχωρος ενός ισχυρά αριθμήσιμα καθοριζόμενου χώρου είναι ισχυρά αριθμήσιμα καθοριζόμενος.
(ii) Αριθμήσιμο γινόμενο ισχυρά αριθμήσιμα καθοριζόμενων χώρων είναι ισχυρά αριθμήσιμα καθοριζόμενος.
(iii) Έστω X τοπολογικός χώρος. Τότε τα ακόλουθα είναι ισοδύναμα:

(1) Ο χώρος X είναι ισχυρά αριθμήσιμα καθοριζόμενος.
(2) Ο χώρος $(\mathcal{K}(X), \tau_\nu)$ είναι ισχυρά αριθμήσιμα καθοριζόμενος.
(3) Ο χώρος $(\mathcal{K}(X), \tau_\nu)$ είναι αριθμήσιμα καθοριζόμενος.

(iv) Συνεχής εικόνα ενός ισχυρά αριθμήσιμα καθοριζόμενου χώρου δεν είναι κατ' ανάγκη ισχυρά αριθμήσιμα καθοριζόμενος.

Μια συνεχής απεικόνιση $f\colon X \to Y$ λέγεται compact covering αν για κάθε συμπαγές υποσύνολο L του Y υπάρχει ένα συμπαγές υποσύνολο K του X τέτοιο ώστε να ισχύει $f(K) = L$.

Αν η απεικόνιση $f\colon X \to Y$ είναι compact covering και ο χώρος X είναι ισχυρά αριθμήσιμα καθοριζόμενος, τότε ο Y είναι ισχυρά αριθμήσιμα καθοριζόμενος.

Ορισμός 5. Ένας χώρος Banach λέγεται *ασθενώς ισχυρά αριθμήσιμα καθοριζόμενος* (SWCD) αν είναι ισχυρά αριθμήσιμα καθοριζόμενος στην ασθενή του τοπολογία.

Ένας χώρος Banach X είναι SWCD αν και μόνον αν ο χώρος (B_X, w) είναι SCD.

Παραδείγματα 6. (i) Κάθε SWKA χώρος Banach είναι SWCD, επομένως και κάθε SWCG χώρος Banach. Ειδικότερα κάθε διαχωρίσιμος χώρος Banach με την ιδιότητα Schur.
(ii) Κάθε διαχωρίσιμος χώρος Banach με διαχωρίσιμο δυϊκό.

Χαρακτηρισμός χώρων Banach SWCD

Πρόταση 7. *Έστω X χώρος Banach. Τότε τα ακόλουθα είναι ισοδύναμα:*

(1) Ο χώρος X είναι SWCD

(2) *Υπάρχει ένας διαχωρίσιμος μετρικός χώρος* M *και μια οικογένεια* $\{W_K : K \in \mathcal{K}(M)\}$ *ασθενώς συμπαγών υποσυνόλων του* X *τέτοια ώστε:*
   - *Αν* $K_1, K_2 \in \mathcal{K}(M)$ *με* $K_1 \subseteq K_2$*, τότε* $W_{K_1} \subseteq W_{K_2}$
   - *Για κάθε* L *ασθενώς συμπαγές υποσύνολο του* X *υπάρχει* $K \in \mathcal{K}(M)$ *τέτοιο ώστε* $L \subseteq W_K$.

Θεώρημα 8. *Έστω* X *διαχωρίσιμος χώρος Banach, ο οποίος δεν περιέχει τον* ℓ^1. *Τότε ο* X^* *είναι διαχωρίσιμος αν και μόνον αν ο* X *είναι SWCD.*

Πόρισμα 9. *Κάθε χώρος Banach* X *SWCD, ο οποίος δεν περιέχει τον* ℓ^1 *είναι Asplund, κατά συνέπεια WCG.*

Αθροίσματα χώρων Banach SWCD

Πρόταση 10. *Έστω* (X_n) *μια ακολουθία χώρων Banach SWCD και* $p \geq 1$. *Τότε ο χώρος* $X = (\sum_{n=1}^{\infty} \oplus X_n)_p$ *είναι SWCD.*

Δε γνωρίζουμε αν ένα c_0- άθροισμα χώρων Banach SWCD είναι SWCD. Έχουμε όμως την ακόλουθη ειδική περίπτωση.

Πρόταση 11. *Έστω* (X_n) *μια ακολουθία διαχωρίσιμων χώρων Banach SWCD και κάθε* X_n *έχει διαχωρίσιμο δυϊκό ή έχει την ιδιότητα Schur. Τότε ο χώρος* $X = (\sum_{n=1}^{\infty} \oplus X_n)_0$ *είναι SWCD.*

Θεώρημα 12. *Για κάθε* $\xi < \omega_1$ *θεωρούμε ένα χώρο Banach* E_ξ *με μια νορμαρισμένη Schauder βάση* $(e_{(n,\xi)})$, *η οποία δεν έχει ασθενώς συγκλίνουσα υπακολουθία. Τότε ο χώρος Banach* $E = (\sum_{\xi<\omega_1} \oplus E_\xi)_p$, *όπου* $p = 0$ *ή* $p > 1$ *είναι WCG, αλλά όχι SWCD.*

Πόρισμα 13. *Ο χώρος Banach* $c_0(\omega_1)$ *δεν είναι SWCD.*

Θεώρημα 14. *Έστω* K *συμπαγής τοπολογικός χώρος. Τότε ο χώρος* $C(K)$ *είναι SWCD αν μόνον αν το* K *είναι αριθμήσιμο.*

Fixed point theorem for three mappings on three complete metric spaces, using implicit relations

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Abstract. A fixed point theorem in three metric spaces is proved. This result extends the results obtained in [3] from two metric spaces to three metric spaces. It generalizes the results obtained in [6,7,8]. A several corollaries are obtained according as the forms of implicit functions.

1. Introduction

In [6], [7] and [3] the following theorems are proved:

Theorem 1 *(Nung)* [6] *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and suppose T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a continuous mapping of Z into X satisfying the inequalities*

$$d(RSTx,RSy)\le c\max\{d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx)\}$$

$$\rho(TRSy,TRz)\le c\max\{\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy)\}$$

$$\sigma(STRz,STx)\le c\max\{\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz)\}$$

for all x in X, y in Y and z in Z, where $0\le c<1$. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu=v, Sv=w$ and $Rw=u$.

Theorem 2 *(Jain et.al.)*[7] *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$d^2(RSy,RSTx)\le c\max\ \{d(x,RSy)\rho(y,Tx),\rho(y,Tx)d(x,RSTx),$$
$$d(x,RSTx)\sigma(Sy,STx),\sigma(Sy,STx)d(x,RSy)\}$$

$$\rho^2(TRz,TRSy)\le c\max\ \{\rho(y,TRz)\sigma(z,Sy),\sigma(z,Sy)\rho(y,TRSy),$$
$$\rho(y,TRSy)d(Rz,RSy),d(Rz,RSy)\rho(y,TRz)\}$$

$$\sigma^2(STx,STRz)\le c\max\ \{\sigma(z,STx)d(x,Rz),d(x,Rz)\sigma(z,STRz),$$
$$\sigma(z,STRz)\rho(Tx,TRz),\rho(Tx,TRz)\sigma(z,STx)\}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v, Sv = w$ and $Rw = u$.

Theorem 3 (Nešić)[3] *Let (X,d) and (Y,ρ) be complete metric spaces. Let T be a mapping of X into Y and S a mapping of Y into X. Denote*

$$M_1(x,y) = \{d^p(x,Sy), \rho^p(y,Tx), \rho^p(y,TSy)\}$$

and

$$M_2(x,y) = \{\rho^p(y,Tx), d^p(x,Sy), d^p(x,STx)\}$$

for all x in X, y in Y and $p = 1,2,3,\ldots$.

Let R^+ be the set of nonnegative real numbers, and let $F_i : R^+ \to R^+$ be a mapping such that $F_i(0) = 0$ and F_i is continuous at 0 for $i = 1,2$.

If T and S satisfying the inequalities

$$\rho^p(Tx,TSy) \le c_1 \max M_1(x,y) + F_1(\min M_1(x,y)),$$

$$d^p(Sy,STx) \le c_2 \max M_2(x,y) + F_2(\min M_2(x,y)),$$

for all x in X and y in Y, where $0 \le c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

2. Main results

We will prove a theorem which generalizes the Theorems Nung [6], Jain, Shrivastava and Fisher [7], Nešić [3] and extends the Theorem Nešić from two to three metric spaces. For this, we will use the implicit functions.

Let $\Phi_4^{(m)}$ be the set of continuous functions with 4 variables

$$\varphi : [0,+\infty)^4 \to [0,+\infty)$$

satisfying the properties:

φ is non descending in respect with each variable.

$\varphi(t,t,t,t) \le t^m, m \in N$.

Denote $I_4 = \{1,2,3,4\}$.

Some examples of such functions are as follows:

Example 4 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$, *with* $m = 1$.

Example 5 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_i t_j : i,j \in I_4\}$, *with* $m = 2$.

Example 6 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1t_2, t_2t_3, t_3t_4, t_4t_1\}$, *with* $m = 2$.

Example 7 $\varphi(t_1,t_2,t_3,t_4)=\max\{t_1^p,t_2^p,t_3^p,t_4^p\}$, *with* $m=p$.

Let Ψ_4 be the set of continuous functions with 4 variables

$$\psi:[0,+\infty)^4\to[0,+\infty)$$

satisfying the property

$$t_1t_2t_3t_4=0\Rightarrow\psi(t_1,t_2,t_3,t_4)=0.$$

Example 8

$$\psi(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3,t_4\}$$
$$\psi(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3\}$$
$$\psi(t_1,t_2,t_3,t_4)=\min\{t_1^p,t_2^p,t_3^p,t_4^p\},etc.$$

Let $\mathcal{F}$ be the set of continuous functions

$$F:[0,+\infty)\to[0,+\infty)$$

with $F(0)=0$ (For example $F(t)=t^k,k>0$).

Theorem 9 *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, such that at least one of them is a continuous mapping. Let $\varphi_i\in\Phi_4^{(m)},\psi_i\in\Psi_4,F_i\in\mathcal{F}$ for $i=1,2,3$. If there exists $q\in[0,1)$ and the following inequalities hold*

$$(1)\qquad d^m(RSy,RSTx)\le q\varphi_1(d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx))+$$
$$+F_1(\psi_1(d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx)).$$

$$(2)\qquad \rho^m(TRz,TRSy)\le q\varphi_2(\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy))+$$
$$+F_2(\psi_2(\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy)).$$

$$(3)\qquad \sigma^m(STx,STRz)\le q\varphi_3(\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz))+$$
$$+F_3(\psi_3(\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz))$$

for all $x\in X,y\in Y$ and $z\in Z$, then RST has a unique fixed point $\alpha\in X$, TRS has a unique fixed point $\beta\in Y$ and STR has a unique fixed point $\gamma\in Z$. Further, $T\alpha=\beta,S\beta=\gamma$ and $R\gamma=\alpha$.

Let $x_0\in X$ be an arbitrary point. We define the sequences $(x_n),(y_n)$ and (z_n) in X,Y and Z respectively as follows:

$$x_n=(RST)^n x_0,y_n=Tx_{n-1},z_n=Sy_n,n=1,2,...$$

Denote

$$d_n=d(x_n,x_{n+1}),\rho_n=\rho(y_n,y_{n+1}),\sigma_n=\sigma(z_n,z_{n+1}),n=1,2,...$$

By the inequality (2), for $y=y_n$ and $z=z_{n-1}$ we get:

$$\rho^m(y_n, y_{n+1}) \le q\varphi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)) +$$
$$+F_2(\psi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)).$$

or

$$\rho_n^m \le q\varphi_2(0, \rho_n, \sigma_{n-1}, d_{n-1}) + F_2(\psi_2(0, \rho_n, \sigma_{n-1}, d_{n-1})) =$$
$$= q\varphi_2(0, \rho_n, \sigma_{n-1}, d_{n-1}) \qquad (4)$$

For the coordinates of the point $(0, \rho_n, \sigma_{n-1}, d_{n-1})$ we have:

$$\rho_n \le \max\{d_{n-1}, \sigma_{n-1}\}, \forall n \in N \ (5)$$

because, in case that $\rho_n > \max\{d_{n-1}, \sigma_{n-1}\}$ for some n, if we replace the coordinates with ρ_n and apply the property (b) of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\rho_n, \rho_n, \rho_n, \rho_n) \le q\rho_n^m.$$

This is impossible since $0 \le q < 1$.

By the inequalities (4), (5) and properties of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}) \le$$
$$\le q \max\{d_{n-1}^m, \sigma_{n-1}^m\}.$$

Thus

$$\rho_n \le \sqrt[m]{q} \max\{d_{n-1}, \sigma_{n-1}\} \ (6)$$

By the inequality (4), for $x = x_{n-1}$ and $z = z_n$ we get:

$$\sigma^m(z_n, z_{n+1}) \le q\varphi_3(\sigma(z_n, z_n), \sigma(z_n, z_{n+1}), d(x_{n-1}, x_n), \rho(y_n, y_{n+1})) +$$
$$+F_3(\psi_3(\sigma(z_n, z_n), \sigma(z_n, z_{n+1}), d(x_{n-1}, x_n), \rho(y_n, y_{n+1})))$$

or

$$\sigma_n^m \le q\varphi_3(0, \sigma_n, d_{n-1}, \rho_n) + F_3(0) =$$
$$= q\varphi_3(0, \sigma_n, d_{n-1}, \rho_n) \qquad (7)$$

In similar way, we get:

$$\sigma_n^m \le q \max\{d_{n-1}^m, \rho_n^m\}, \forall n \in N.$$

By this inequality and (6) we get:

$$\sigma_n \le \sqrt[m]{q} \max\{d_{n-1}, \sigma_{n-1}\}, \forall n \in N \ (8)$$

By (1) for $x = x_n$ and $y = y_n$ we get:

$$d^m(x_n, x_{n+1}) \le q\varphi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})) +$$
$$+F_1(\psi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})))$$

or

$$d_n^m \le q\varphi_1(0, d_n, \rho_n, \sigma_n) + F(0) =$$
$$= q\varphi_1(0, d_n, \rho_n, \sigma_n) \qquad (9)$$

For the same reasons we used to (5), for the coordinates of the point $(0, d_n, \rho_n, \sigma_n)$ we have:

$$d_n \le \max\{\rho_n, \sigma_n\}, \forall n \in N.$$

Applying to (9) the properties of φ_1 and the inequalities (6), (8) we get:

$$d_n \le \sqrt[m]{q} \max\{\rho_n, \sigma_n\} \le \sqrt[m]{q}(\sqrt[m]{q} \max\{d_{n-1}, \sigma_{n-1}\}) =$$
$$= \sqrt[m]{q}(\sqrt[m]{q}) \max\{d_{n-1}, \sigma_{n-1}\} \le \sqrt[m]{q} \max\{d_{n-1}, \sigma_{n-1}\}$$

or

$$d_n \le \sqrt[m]{q} \max\{d_{n-1}, \sigma_{n-1}\} \quad (10)$$

By the inequalities (6), (8) and (10), using the mathematical induction, we get:

$$d(x_n, x_{n+1}) \le r^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\rho(y_n, y_{n+1}) \le r^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\sigma(z_n, z_{n+1}) \le r^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$

where $\sqrt[m]{q} = r < 1$.

Thus the sequences $(x_n), (y_n)$ and (z_n) are Cauchy sequences. Since the metric spaces $(X, d), (Y, \rho)$ and (Z, σ) are complete metric spaces we have:

$$\lim_{n\to\infty} x_n = \alpha \in X, \lim_{n\to\infty} y_n = \beta \in Y, \lim_{n\to\infty} z_n = \gamma \in Z.$$

Assume that S is a continuous mapping. Then by

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n.$$

it follows

$$S\beta = \gamma. \quad (11)$$

By (1), for $y = \beta$ and $x = x_n$ we get:

$$d^m(RS\beta, x_{n+1}) \le q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta)) +$$
$$+ F_1(\psi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta))).$$

By this inequality and (11) we get:

$$d^m(RS\beta, x_{n+1}) \le q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), 0) +$$
$$+ F_1(0).$$

Letting n tend to infinity, we get

$$d^m(RS\beta, \alpha) \le q\varphi_1(d(RS\beta, \alpha), 0, 0, 0) \le q d^m(RS\beta, \alpha)$$

or

$$d(RS\beta, \alpha) = 0 \Leftrightarrow RS\beta = \alpha. \quad (12)$$

By (2), for $z = S\beta$ and $y = y_n$ we get:

$$\rho^m(TRS\beta, y_{n+1}) \le q\varphi_2(\rho(y_n,TRS\beta),\rho(y_n,y_{n+1}),\sigma(S\beta,z_n),d(x_n,RS\beta))+$$
$$+F_2(\psi_2(\rho(y_n,TRS\beta),\rho(y_n,y_{n+1}),\sigma(S\beta,z_n),d(x_n,RS\beta))).$$

Letting n tend to infinity and using (11), (12) we get:

$$\rho^m(TRS\beta,\beta) \le q\varphi_2(\rho(\beta,TRS\beta),0,0,0)+F(0).$$

or

$$\rho^m(TRS\beta,\beta) \le q\rho^m(\beta,TRS\beta) \Leftrightarrow TRS\beta=\alpha. \quad (13)$$

By (11), (12) and (13) it follows:

$$TRS\beta = TR\gamma = T\alpha = \beta$$
$$STR\gamma = ST\alpha = S\beta = \gamma$$
$$RST\alpha = RS\beta = \beta\gamma = \alpha$$

Thus, we proved that the points α,β,γ are fixed points of RST, TRS and STR respectively.

In the same conclusion we would arrive if one of the mappings R or T would be continuous.

Let we prove now the iniquity of the fixed points α,β and γ.

Assume that there is α' a fixed point of RST different from α.

By (1) for $x=\alpha'$ and $y=T\alpha$ we get:

$$d^m(\alpha,\alpha') = d^m(RST\alpha,RST\alpha') \le$$
$$\le q\varphi_1(d(\alpha',RST\alpha),d(\alpha',RST\alpha'),\rho(T\alpha,T\alpha'),\sigma(ST\alpha,ST\alpha'))+$$
$$+F_1(\psi_1(d(\alpha',RST\alpha),d(\alpha',RST\alpha'),\rho(T\alpha,T\alpha'),\sigma(ST\alpha,ST\alpha'))) =$$
$$= q\varphi_1(d(\alpha',\alpha),0,\rho(T\alpha,T\alpha'),\sigma(ST\alpha,ST\alpha'))+F(0) \le$$
$$\le q\max\{d^m(\alpha',\alpha),\rho^m(T\alpha,T\alpha'),\sigma^m(ST\alpha,ST\alpha')\}$$

or

$$d^m(\alpha,\alpha') = q\max A \quad (14)$$

where $A=\{d^m(\alpha',\alpha);\rho^m(T\alpha,T\alpha');\sigma^m(ST\alpha,ST\alpha')\}$.

We distinguish the following three cases:

Case I: If $\max A = d^m(\alpha',\alpha)$, then the inequality (14) implies

$$d^m(\alpha,\alpha') \le qd^m(\alpha',\alpha) \Leftrightarrow \alpha'=\alpha.$$

Case II: If $\max A = \rho^m(T\alpha,T\alpha')$, then the inequality (14) implies

$$d^m(\alpha,\alpha') \le q\rho^m(T\alpha,T\alpha') \quad (15)$$

Continuing our argumentation for the Case 2, by (2) for $z=ST\alpha$ and $y=T\alpha'$ we have:

$$\rho^m(T\alpha,T\alpha') = \rho^m(TRST\alpha,TRST\alpha') \le$$
$$\le q\varphi_2(\rho(T\alpha',TRST\alpha),\rho(T\alpha',TRST\alpha'),\sigma(ST\alpha,ST\alpha'),d(RST\alpha',RST\alpha))$$
$$+F_2(\psi_2(\rho(T\alpha',TRST\alpha),\rho(T\alpha',TRST\alpha'),\sigma(ST\alpha,ST\alpha'),d(RST\alpha',RST\alpha))) =$$
$$= q\varphi_2(\rho(T\alpha',T\alpha),0,\sigma(ST\alpha,ST\alpha'),d_1(\alpha,\alpha'))+F(0) =$$
$$\le q\max A \qquad (16)$$

Since in Case II, $\max A = \rho^m(T\alpha,T\alpha')$, by (16) it follows

$$\rho^m(T\alpha,T\alpha') \le q\rho^m(T\alpha,T\alpha')$$

or

$$\rho(T\alpha,T\alpha') = 0.$$

By (15), it follows $d(\alpha,\alpha') = 0$.

Case III: If $\max A = \sigma^m(ST\alpha,ST\alpha')$, then by (14) it follows

$$d^m(\alpha,\alpha') \le q\sigma^m(ST\alpha,ST\alpha') \quad (17)$$

By the inequality (3), for $x = RST\alpha, z = ST\alpha'$, in similar way we obtain:

$$\sigma^m(ST\alpha,ST\alpha') \le q\max A = q\sigma^m(ST\alpha,ST\alpha')$$

It follows

$$\sigma(ST\alpha,ST\alpha') = 0$$

and by (17) it follows

$$d(\alpha,\alpha') = 0.$$

Thus, we have again $\alpha = \alpha'$.

In the same way, it is proved the nicety of β and γ.

We emphasize the fact that it is necessary the continuity of at least one of the mappings T,S and R. The following example shows this.

Example 10 *Let* $X = Y = Z = [0,1]; d = \rho = \sigma$ *such that* $d(x,y) = |x-y|, \forall x,y \in [0,1]$. *We consider the mappings* $T,S,R:[0,1] \to [0,1]$ *such that*

$$Tx = Rx = Sx = \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{x}{2} & \text{for } x \in (0,1] \end{cases}$$

We have

$$STx = RSx = TRx = \begin{cases} \frac{1}{2} & for\ x = 0 \\ \frac{x}{4} & for\ x \in (0,1] \end{cases}$$

and

$$RSTx = TRSx = STRx = \begin{cases} \frac{1}{4} & for\ x = 0 \\ \frac{x}{8} & for\ x \in (0,1] \end{cases}$$

We observe that the inequalities (1), (2) and (3) are satisfied for $\varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(1)}$ with $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$, where $q = \frac{1}{2}$ and $F = 0$. It can be seen that none of the mappings RST, TRS, STR has a fixed point. This is because none of the mappings T, R, S is a continuous mapping.

3. Corollaries

Corollary 3.1 *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, such that at least one of them is a continuous mapping. Let $F:[0,+\infty) \to [0,+\infty)$ be continuous with $F(0)=0$. If there exists $q \in [0,1)$ and $m \in N$ such that the following inequalities hold*

(1) $$d^m(RSy, RSTx) \le q \max(d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx)) + \\ + F_1(\min(d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx)).$$

(2) $$\rho^m(TRz, TRSy) \le q \max(\rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy)) + \\ + F_2(\min(\rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy)).$$

(3) $$\sigma^m(STx, STRz) \le q \max(\sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz)) + \\ + F_3(\min(\sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz))$$

for all $x \in X, y \in Y$ and $z \in Z$, then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$ and STR has a unique fixed point $\gamma \in Z$. Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

The proof follows by Theorem 2.6 in the case $F_1 = F_2 = F_3 = F, \varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(m)}$ such that $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1^m, t_2^m, t_3^m, t_4^m\}$ and $\psi_1 = \psi_2 = \psi_3 = \psi$, where $\psi(t_1,t_2,t_3,t_4) = \min\{t_1^m, t_2^m, t_3^m, t_4^m\}$.

Corollary 3.1 extends Theorem 1.3 (Nešić [3]) from two in three metric spaces.

Corollary 3.2 *Theorem 1.1 (Nung [6]) is taken by Corollary 3.1 for* $m=1$ *and* $F=0$.

Corollary 3.3 *Theorem 1.2 (Jain et. al. [7]) is taken by Theorem 2.6 in case* $F_1 = F_2 = F_3 = 0; \varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(2)}$ *such that* $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1t_3, t_2t_3, t_2t_4, t_4t_1\}$.

Corollary 3.4 *Theorem Kikina (Theorem 2.1, [8]) is taken by Corollary 3.1 in case* $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1^m, t_2^m, t_3^m\}$ *and* $\psi(t_1,t_2,t_3,t_4) = \min\{t_1^m, t_2^m, t_3^m\}$.

Corollary 3.5 *Let* $(X,d),(Y,\rho)$ *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z. $\varphi_i \in \Phi_3, F_i \in \mathbb{F}$ *for* $i=1,2$. *If there exists* $q\in[0,1)$ *such that the following inequalities hold*

$$(1') \qquad d(Sy,STx) \le q\varphi_1(d(x,Sy),d(x,STx),\rho(y,Tx)) + F_1(\psi_1(d(x,Sy),d(x,STx),\rho(y,Tx)).$$

$$(2') \qquad \rho^m(Tx,TSy) \le q\varphi_2(\rho(y,Tx),\rho(y,TSy),d(x,Sy)) + F_2(\psi_2(\rho(y,Tx),\rho(y,TSy),d(x,Sy)).$$

for all $x\in X, y\in Y$, then ST has a unique fixed point $\alpha\in X$ and TS has a unique fixed point $\beta\in Y$. Further, $T\alpha=\beta, S\beta=\gamma$.

By Theorem 2.6, if we take: $Z=X, \sigma=d$ the mapping R as the identity mapping in X, $\varphi_i(t_1,t_2,t_3,t_4)=\varphi_i(t_1,t_2,t_3), \psi_i(t_1,t_2,t_3,t_4)=\psi_i(t_1,t_2,t_3)$, then the inequality (1) takes the form (1'), the inequality (2) takes the form (2') and the inequality (3) is always satisfied since his left side is $\sigma^m(STx,STx)=0$. Thus, the satisfying of the conditions (1), (2) and (3) is reduced in satisfying of the conditions (1') and (2').

The mappings T and S may be not continuous, while fromt he mappings T,S and R for which we applied Theorem 2.6, the identity mapping R is continuous. This completes the proof.

We have the following corollary.

Corollary 3.6 *(Theorem Nešić [3]). Theorem 1.3 is taken by Corollary 3.5 for* $\varphi_1=\varphi_2=\varphi; \psi_1=\psi_2=\psi$ *such that* $\varphi_1(t_1,t_2,t_3)=\max\{t_1^m,t_2^m,t_3^m\}$ *and* $\psi(t_1,t_2,t_3)=\min\{t_1^m,t_2^m,t_3^m\}$.

We emphasize the fact that in the Theorem 1.3, the mappings F_1 and F_2 can be replaced by $F(t)=\max\{F_1(t),F_2(t)\}$ and c_1,c_2 can be replaced by $q=\max\{c_1,c_2\}$.

Corollary 3.7 *Theorem Popa (Theorem 2, [2]) is taken by Corollary 3.5 for*

$\varphi_1 = \varphi_2 = \varphi$ *such that* $\varphi(t_1, t_2, t_3) = \max\{t_1t_2, t_1t_3, t_2t_3\}$ *with* $m = 2$ *and* $F = 0$.

We also emphasize here that the constants c_1, c_2 can be replaced by $q = \max\{c_1, c_2\}$.

Remark. As corollaries of these results we can obtain other propositions determined by the form of implicit functions, for example Proposition Popa (Corollary 2, [2]), Theorem Fisher (Theorem 1, [1]) etc.

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Brownian motion on Spheres

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Abstract

We evaluate explicity certain quantities regarding the Brownian motion process on the n-dimensional sphere of radius a. We start with the transition densities of the process. Then we calculate some probabilistic quantities (e.g. moments) of the hitting times of specific symmetric domains.

1 Definitions and properties

1.1 The n-sphere

Definition 1.1 *Let $n \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$. The n-dimensional sphere S^n with center $(c_1, ..., c_{n+1})$ and radius $a > 0$ is defined to be the set of all points $x = (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1}$ satisfying $(x_1 - c_1)^2 + \cdots + (x_{n+1} - c_{n+1})^2 = a^2$. Thus,*

$$S^n = \{\ (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid (x_1 - c_1)^2 + \cdots + (x_{n+1} - c_{n+1})^2 = a^2\ \}$$

There are two different coordinate systems we use, the stereographic projection coordinates and the spherical coordinates

Definition 1.2 *We consider $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ to be the plane given by $x^{n+1} = 0$. For convenience, we will let $(x_1, x_2, ..., x_n, x_{n+1})$ be coordinates on $\mathbb{R}^{n+1}$ and $(\xi_1, \xi_2, ..., \xi_n)$ be coordinates on $\mathbb{R}^n \subset \mathbb{R}^{n+1}$.*
Let $S^n = \{\ (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 + (x_{n+1} - a)^2 = a^2\ \}$. The stereographic projection coordinates of S^n is the map $\Phi : S^n - \{0, 0, \ldots, 2a\} \to \mathbb{R}^n$ given by

$$\Phi\,(x_1, x_2, ..., x_n, x_{n+1}) = \left(\frac{2ax_1}{2a - x_{n+1}}, ..., \frac{2ax_n}{2a - x_{n+1}}\right).$$

This map defines coordinates $(\xi_1, \xi_2, ..., \xi_n)$ on S^n so that the point $(x_1, x_2, ..., x_n, x_{n+1})$ of S^n has coordinates $(\xi_1, \xi_2, ..., \xi_n)$ where,

$$\xi_1 = \frac{2ax_1}{2a - x_{n+1}} \quad \ldots, \; \xi_n = \frac{2ax_n}{2a - x_{n+1}}.$$

The inverse map is given by

$$x_1 = \frac{4a^2\xi_1}{\xi_1^2+\cdots+\xi_n^2+4a^2}, \ \ldots, \ x_n = \frac{4a^2\xi_n}{\xi_1^2+\cdots+\xi_n^2+4a^2}, \ x_{n+1} = \frac{2a\left(\xi_1^2+\cdots+\xi_n^2-1\right)}{\xi_1^2+\cdots+\xi_n^2+4a^2}.$$

Definition 1.3 *The points of the n-sphere with center at the origin and radius a may also be described in spherical coordinates in the following way:*

$$x_1 = a\cos\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}\sin\theta_n$$

$$x_2 = a\sin\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}\sin\theta_n$$

$$x_k = a\cos\theta_{k-1}\sin\theta_k \ldots \sin\theta_{n-1}\sin\theta_n \quad \textit{for} \quad k = 3,4,\ldots,n$$

$$x_{n+1} = a\cos\theta_n,$$

where

$$0 \le \theta_1 < 2\pi, \quad \textit{and} \quad 0 \le \theta_i \le \pi \quad \textit{for} \quad i = 2,3,\ldots,n.$$

From now on θ_n will be denoted by φ.

1.2 The Laplace-Beltrami operator

Using the spherical coordinates, the Laplace-Beltrami operator of a smooth function f on S^n is

$$\Delta_n f = \frac{1}{\sqrt{\det(g)}} \cdot \sum_{i=1}^{n} \frac{\partial}{\partial\theta_i}\left(\sqrt{\det(g)} \cdot \sum_{j=1}^{n} g^{ij}\frac{\partial f}{\partial\theta_j}\right), \tag{1.1}$$

where

$$\det(g) = a^{2n}\prod_{k=2}^{n}(\sin\theta_k)^{2(k-1)}$$

$$g^{ij} = 0, \text{ if } i \neq j \quad g^{ii} = \frac{1}{a^2\sin^2\theta_{i+1}\cdot\ldots\cdot\sin^2\theta_n} \quad \text{and } \theta_n = \varphi.$$

If f is independent of $\theta_1, \theta_2, \ldots, \theta_{n-1}$, the Laplace Beltrami operator of f is

$$\Delta_n f = \frac{1}{a^2}\left((n-1)\cot\varphi \cdot \frac{\partial f}{\partial\varphi} + \frac{\partial^2 f}{\partial\varphi^2}\right). \tag{1.2}$$

Example 1.1 *Using the spherical coordinates. If $M = S^2$, i.e.*

$$S^2 = \{\, x = (a\cos\theta\sin\varphi, a\sin\theta\sin\varphi, a\cos\varphi) \in \mathbb{R}^3 \mid 0 \le \theta < 2\pi, \ 0 \le \varphi \le \pi\}$$

We have:

$$x_\theta = \frac{\partial x}{\partial\theta} = (-a\sin\theta\sin\varphi, a\cos\theta\sin\varphi, 0)$$

$$x_\varphi = \frac{\partial x}{\partial\varphi} = (a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, -a\sin\varphi)$$

$$g = [g_{ij}] = \begin{pmatrix} x_\theta x_\theta & x_\theta x_\varphi \\ x_\varphi x_\theta & x_\varphi x_\varphi \end{pmatrix}$$

i.e.

$$g = [g_{ij}] = \begin{pmatrix} a^2 \sin^2 \varphi & 0 \\ 0 & a^2 \end{pmatrix}$$

and

$$g^{-1} = [g^{ij}] = \begin{pmatrix} \frac{1}{a^2 \sin^2 \varphi} & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix}.$$

Hence the Laplace-Beltrami operator of a smooth function f on S^2 is

$$\Delta_2 f = \frac{1}{a^2 \sin \varphi} \left(\frac{f_{\theta\theta}}{\sin \varphi} + f_\varphi \cos \varphi + f_{\varphi\varphi} \sin \varphi \right). \tag{1.3}$$

In the case where the function f is independent of θ the Laplace-Beltrami operator of f is

$$\Delta_2 f = \frac{1}{a^2 \sin \varphi} \left(f_\varphi \cos \varphi + f_{\varphi\varphi} \sin \varphi \right). \tag{1.4}$$

Similarly the Laplace-Beltrami operator of a smooth function f on S^3 is

$$\Delta_3 f = \frac{1}{a^2 \sin^2 \theta_2 \sin^2 \varphi} \cdot \frac{\partial^2 f}{\partial \theta_1^2} + \frac{1}{a^2 \sin \theta_2 \sin^2 \varphi} \cdot \frac{\partial}{\partial \theta_2} \left(\sin \theta_2 \frac{\partial f}{\partial \theta_2} \right) + \frac{1}{a^2 \sin^2 \varphi} \cdot \frac{\partial}{\partial \varphi} \left(\sin^2 \varphi \frac{\partial f}{\partial \varphi} \right). \tag{1.5}$$

1.3 Brownian Motion on a compact and smooth Riemmanian Manifold M

Definition 1.4 *Let M be a compact and smooth Riemannian manifold and Δ its corresponding Laplace-Beltrami operator. The unique solution of the differential equation*

$$\frac{\partial P}{\partial t} - \frac{1}{2} \Delta_x P = 0, \tag{1.6}$$

where Δ_x is Δ acting on the x-variables and the initial condition

$$P(t, x, y) \to \delta_x(y) \quad as, \quad t \to 0^+ \tag{1.7}$$

(where $\delta_x(y)$ is the delta mass at $x \in M$) is called the heat kernel on M.

It has been proved by J.Dodziak [3] that the heat kernel always exists and is smooth in (t, x, y). Moreover the heat kernel possesses the following properties.

1. Symmetry in x, y, that is

$$P(t, x, y) = P(t, y, x)$$

2. The semigroup identity: For any $s \in (0, t)$

$$P(t, x, y) = \int_M P(s, x, z) P(t - s, z, y) d\mu(z)$$

where $d\mu$ is the area measure element of M.

3. The total mass equality: For all $t > 0$ and $x \in M$

$$\int_M P(t, x, y) d\mu(y) = 1. \tag{1.8}$$

4. As $t \to \infty$, $p(t, x, y)$ approaches the uniform density on M, i.e. $p(t, x, y) \to c$ where,

$$c = \frac{1}{Area(M)}.$$

Definition 1.5 *The Brownian motion X_t, $t \geq 0$, on a Riemannian manifold M is a Markov process with transition density function $P(t, x, y)$, the heat kernel associated with the Laplace-Beltrami operator.*

The case of S^n

In the case where $M = S^n$, $n \geq 2$, the transition density function $P(t, x, y)$ of the Brownian motion X_t depends only on t and $d(x, y)$, the distance between x and y. Thus in spherical coordinates it depends on t and the angle φ between x and y. Hence the transition density function of the Brownian motion can be written as

$$P(t, x, y) = p(t, \varphi) \tag{1.9}$$

where $p(t, \varphi)$ is the smallest positive solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta_n p = \frac{1}{2a^2}\left((n-1)\cot\varphi \cdot \frac{\partial p}{\partial \varphi} + \frac{\partial^2 p}{\partial \varphi^2}\right) \tag{1.10}$$

and

$$\lim_{t \to 0+} aA_{n-1} p(t, \varphi) \cdot \sin^{n-1}(\varphi) = \delta(\varphi). \tag{1.11}$$

Here δ is the Dirac delta function and A_n denotes the area of the n-dimensional sphere S^n with radius a.It is well known that

$$A_n = \frac{2\pi^{\frac{n+1}{2}} a^n}{\Gamma(\frac{n+1}{2})}, \tag{1.12}$$

where $\Gamma(x)$ is the Gamma function. More precisely

$$A_n = \frac{2\pi^{\frac{n+1}{2}} a^n}{(\frac{n-1}{2})!} \qquad \text{for n odd} \tag{1.13}$$

$$A_n = \frac{2^n(\frac{n}{2}-1)!\pi^{\frac{n}{2}} a^n}{(n-1)!} \qquad \text{for n even} \tag{1.14}$$

Remark 1.1 *The fact that S^n is a compact and smooth manifold implies that (1.10) - (1.11) has a unique positive solution which also satisfies*

$$\int_{S^n} p(t, x, y) d\mu(y) = 1. \tag{1.15}$$

Furthermore, as $t \to \infty$, $p(t, x, y)$ approaches the uniform density on S^n, i.e. $p(t, x, y) \to c$ where,

$$c = \frac{1}{A_n}.$$

Lemma 1.1 *If $s > 0$ and $z, \gamma \in \mathbb{C}$, then*

$$\sum_{n\in\mathbb{Z}} \exp\left(-s^2(z+n)^2 - i\gamma n\right) = \frac{\sqrt{\pi}}{s}\exp\left(i\gamma z - \frac{\gamma^2}{4s^2}\right)\sum_{n\in\mathbb{Z}}\exp\left(\frac{-\pi^2 n^2}{s^2} - \frac{\pi\gamma n}{s^2} + 2\pi i z n\right) \tag{1.16}$$

Proof

Poisson summation formula is applied to the function

$$f(x) = \exp\left(-Ax^2 + Bx\right), \quad A > 0, \quad B \in \mathbb{C}$$

■

2 Transition density function of the Brownian motion on S^1, S^2 and S^3

2.1 The Case of S^1

Let X_t, $t \geq 0$ be the Brownian motion on a 1-dimensional sphere S^1 of radius a.The transition density function $p(t, \varphi)$ of X_t is the unique solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta_1 p \tag{2.1}$$

and

$$\lim_{t\to 0+} a \cdot p(t, \varphi) = \delta(\varphi) \tag{2.2}$$

Here Δ_1 is the Laplace-Beltrami operator on S^1. Therefore we have that $p(t, \varphi)$ is the unique solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2a^2}\frac{\partial^2 p(t,\varphi)}{\partial \varphi^2} \tag{2.3}$$

and

$$\lim_{t\to 0+} a \cdot p(t, \varphi) = \delta(\varphi) \tag{2.4}$$

Proposition 2.1 *The transition density function of the Brownian motion $X_t, t \geq 0$ on S^1 with radius a is the function*

$$p(t, \varphi) = \frac{1}{2\pi a}\sum_{n\in\mathbb{Z}}\exp\left(-\frac{n^2 t}{2a^2} + in\varphi\right). \tag{2.5}$$

Remark 2.1 *The function (2.5) can be also expressed in the following ways*

$$p(t, \varphi) = \frac{1}{\pi a}\sum_{n\in\mathbb{N}}\left[\exp\left(-\frac{n^2 t}{2a^2}\right)\cos(n\varphi)\right] - \frac{1}{2\pi a} \tag{2.6}$$

and

$$p(t, \varphi) = \frac{1}{\sqrt{2\pi t}}\sum_{n\in\mathbb{Z}}\exp\left(-\frac{a^2}{2t}(\varphi - 2\pi n)^2\right). \tag{2.7}$$

2.2 The Case of S^2

Let X_t, $t \geq 0$ be the Brownian motion on a 2-dimensional sphere S^2 of radius a. From the (1.10), (1.11) and (1.13) the transition density function $p(t, \varphi)$ of X_t is the unique solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2a^2 \sin \varphi}\left(\frac{\partial^2 p(t,\varphi)}{\partial \varphi^2} \sin \varphi + \frac{\partial p}{\partial \varphi} \cos \varphi\right) \tag{2.8}$$

and

$$\lim_{t \to 0+} 2\pi a^2 \sin \varphi \cdot p(t, \varphi) = \delta(\varphi). \tag{2.9}$$

The solution to the diffusion equation

$$\frac{\partial K(t,\varphi)}{\partial t} = \frac{1}{\sin \varphi}\left(\cos \varphi \frac{\partial K(t,\varphi)}{\partial \varphi} + \sin \varphi \frac{\partial^2 K(t,\varphi)}{\partial \varphi^2}\right) \tag{2.10}$$

with initial condition

$$\lim_{t \to 0+} 2\pi \sin(\varphi) K(t, \varphi) = \delta(\varphi) \tag{2.11}$$

is given by the function

$$K(t,\varphi) = \frac{1}{4\pi} \sum_{n \in \mathbb{N}} (2n+1) \exp\left(-n(n+1)\sqrt{2t}\right) P_n^0(\cos \varphi) \tag{2.12}$$

see [2].
Here P_n^0 is the associated Legendre polynomials of order zero, i.e.

$$P_n^0(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1) \tag{2.13}$$

This fact implies the following

Proposition 2.2 *The transition density function of the Brownian motion X_t, $t \geq 0$ on S^2 with radius a it is given by the function*

$$p(t,\varphi) = \frac{1}{4\pi a^2} \sum_{n \in \mathbb{N}} (2n+1) \exp\left(-\frac{n(n+1)\sqrt{t}}{a}\right) P_n^0(\cos \varphi) \tag{2.14}$$

2.3 The Case of S^3

Let X_t, $t \geq 0$ be the Brownian motion on a 3-dimensional sphere S^3 of radius a. From the (1.10), (1.11) and (1.14) the transition density function $p(t, \varphi)$ of X_t is the unique solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2a^2}\left(\frac{\partial^2 p}{\partial \varphi^2} + 2 \cot \varphi \frac{\partial p}{\partial \varphi}\right) \tag{2.15}$$

and

$$\lim_{t \to 0+} 4\pi a^3 \cdot \sin^2(\varphi) p(t, \varphi) = \delta(\varphi). \tag{2.16}$$

The function $p(t, \varphi)$ satisfies:

1.

$$p(t,\varphi) > 0 \qquad \text{for every} \qquad (t,\varphi) \in \mathbb{R}^+ \times [o,\pi]$$

2.

$$4\pi a^3 \int_0^\pi p(t,\varphi)\sin^2\varphi d\varphi = 1$$

3.

$$p(t,\varphi) \to \frac{1}{A_3}, \quad \text{as} \quad t \to \infty$$

where

$$A_3 = 2\pi^2 a^3.$$

Proposition 2.3 *The transition density function of the Brownian motion $X_t, t \geq 0$ on S^3 with radius a is the function*

$$p : \mathbb{R}^+ \times (0,\pi) \to \mathbb{R},$$

with

$$p(t,\varphi) = \frac{\exp\left(\frac{t}{2a^2}\right)(2t\pi)^{-\frac{3}{2}}}{\sin\varphi} \sum_{n\in\mathbb{Z}} (\varphi + 2n\pi)\exp\left(-\frac{(\varphi+2n\pi)^2 a^2}{2t}\right) \tag{2.17}$$

Proposition 2.4 *If $\varphi \in (0,\pi)$ The function (2.17) can be also expressed in the following ways*

$$p(t,\varphi) = -\frac{i}{4\pi^2 a^3 \sin\varphi} \sum_{n\in\mathbb{Z}} n \exp\left(-\frac{t(n^2-1)}{2a^2} + i\varphi n\right) \tag{2.18}$$

and

$$p(t,\varphi) = \frac{1}{2\pi^2 a^3 \sin\varphi} \sum_{n\in\mathbb{N}} n\sin(n\varphi)\exp\left(-\frac{t(n^2-1)}{2a^2}\right) \tag{2.19}$$

Remark 2.2 *Formula (2.19) implies that $p(t,\varphi)$ is analytic at $\varphi = 0$ and $\varphi = \pi$ and*

$$p(t,0) = \lim_{\varphi\to 0^+} p(t,\varphi) = \frac{1}{2\pi^2 a^3} \sum_{n\in\mathbb{N}} n^2 \exp\left(-\frac{t(n^2-1)}{2a^2}\right) \tag{2.20}$$

and

$$p(t,\pi) = \lim_{\varphi\to \pi^-} p(t,\varphi) = \frac{1}{2\pi^2 a^3} \sum_{n\in\mathbb{N}} n^2(-1)^n \exp\left(-\frac{t(n^2-1)}{2a^2}\right). \tag{2.21}$$

Reminder

The ϑ_3 function of Jacobi is

$$\vartheta_3(z,r) = 1 + 2\sum_{\mathbb{N}} \exp\left(i\pi r n^2\right)\cos(2nz), \tag{2.22}$$

where $r \in \mathbb{C}$ with $Im\{r\} > 0$.
It follows that

$$p(t,\varphi) = -\frac{1}{4\pi^2 a^3 \sin\varphi} \exp\left(\frac{t}{2a^2}\right) \frac{\partial\vartheta_3\left(\frac{\varphi}{2}, \frac{ti}{2a^2\pi}\right)}{\partial\varphi} \tag{2.23}$$

3 Exit times

We recall some basic definitions.

Definition 3.1 *Let us consider a measurable space $\{\Omega, \mathcal{F}\}$ equipped with a filtration $\{\mathcal{F}_t\}$. A random variable T is a stopping time with respect to the filtration $\{\mathcal{F}_t\}$, if for every $t \geq 0$*

$$\{\omega \in \Omega | \, T(\omega) \leq t\} \in \mathcal{F}_t.$$

Let X_t be the Brownian motion in S^n and $D \subset S^n$ a domain. Then

$$T = \inf\{\, t \geq 0 | \, X_t \notin D\}$$

is a stopping time with respect to $\mathcal{F}_t = \sigma\{\, X_s | \, 0 \leq s \leq t\}$, called the exit time of D.

3.1 Expectations of exit times on S^n

Proposition 3.1 *Let $\varphi_1, \varphi_2 \in (0, \pi)$, such that $\varphi_1 < \varphi_2$, both fixed. We consider the set D in S^1, such that*

$$D = (\varphi_1, \varphi_2)\,.$$

Of course,

$$\partial D = \{\varphi_1, \varphi_2\}\,.$$

If X_t is the Brownian motion on S^1 starting at the point

$$\varphi \in D,$$

then the expectation of T is given by

$$E^{\varphi}[T] = -a^2\,(\varphi - \varphi_1)\,(\varphi - \varphi_2)\,. \tag{3.1}$$

Proof.

It's known that, (see [4]),the function $E^{\varphi}[T]$ satisfies the Poisson equation on D with Dirichlet boundary data. By the unique solution of this equation solution

$$u(\varphi) = E^{\varphi}[T]$$

is the unique solution of the differential equation

$$\frac{1}{2}\Delta_1 u = -1, \tag{3.2}$$

with boundary condition

$$u(\varphi_1) = u(\varphi_2) = 0. \tag{3.3}$$

Hence

$$u(\varphi) = -a^2\,(\varphi - \varphi_1)\,(\varphi - \varphi_2)\,.$$

■

Proposition 3.2 *Let $\varphi_0 \in (0, \pi)$ be fixed. We consider the set D in $S^n, n \geq 2$, such that*

$$D = \{(\theta_1, \ldots, \theta_{n-1}, \varphi) | \, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad for \quad i = 2, \ldots, n-1 \quad and \quad \varphi \in [0, \varphi_0)\}.$$

Of course,

$$\partial D = \{(\theta_1, \ldots, \theta_{n-1}, \varphi) | \, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad for \quad i = 2, \ldots, n-1 \quad and \quad \varphi = \varphi_0\}.$$

If X_t is the Brownian motion on S^n starting at the point

$$A = (\theta_1, \ldots, \theta_{n-1}, \varphi) \in D,$$

then the expectation of T is given by

$$E^A[T] = 2a^2 \int_{\varphi}^{\varphi_0} \frac{\int_0^x (\sin \omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx. \tag{3.4}$$

Proof.

It is known that [4],

$$u(\theta_1, \ldots; \theta_{n-1}, \varphi) = E^A[T]$$

is the unique solution of the differential equation

$$\frac{1}{2} \Delta_n u = -1, \tag{3.5}$$

with boundary condition

$$u(\theta_1, \ldots, \theta_{n-1}, \varphi_0) = 0. \tag{3.6}$$

Here Δ_n is the Laplace-Beltrami operator on S^n.
By the symmetry of D, it follows that the expectation value of T is independent of $\theta_1, \ldots, \theta_{n-1}$. Hence u is independent of θ_i, for $i = 1, \ldots, n-1$. The differential equation (3.5) takes the form

$$\frac{1}{2a^2} \left[(n-1) \cot(\varphi) \frac{du}{d\varphi} + \frac{d^2 u}{d\varphi^2} \right] = -1, \tag{3.7}$$

with boundary condition

$$u(\varphi_0) = 0. \tag{3.8}$$

Set

$$f(\varphi) = \frac{du}{d\varphi}$$

Hence from (3.7)

$$\frac{1}{2a^2} \left[(n-1) \cot(\varphi) f(\varphi) + \frac{df(\varphi)}{d\varphi} \right] = -1$$

or

$$(n-1) \cos(\varphi) f(\varphi) + \sin(\varphi) \frac{df(\varphi)}{d\varphi} = -2a^2 \sin(\varphi),$$

multiplying by $(\sin\varphi)^{n-2}$ implies that

$$(n-1)(\sin\varphi)^{n-2}\cos(\varphi)f(\varphi)+(\sin\varphi)^{n-1}\frac{df(\varphi)}{d\varphi}=-2a^2(\sin\varphi)^{n-1}.$$

Thus

$$f(\varphi)=-\frac{2a^2}{(\sin\varphi)^{n-1}}\int_0^{\varphi}(\sin\omega)^{n-1}d\omega+\frac{c_1}{(\sin\varphi)^{n-1}}.$$

Therefore

$$u(\varphi)=-2a^2\int_0^{\varphi}\frac{\int_0^x(\sin\omega)^{n-1}d\omega}{(\sin x)^{n-1}}dx+c_1\int_0^{\varphi}\frac{1}{(\sin x)^{n-1}}dx+c_2 \tag{3.9}$$

However

$$\varphi\in(0,\varphi_0)\quad,\quad u(\varphi)=E^A[T]<\infty.$$

Hence

$$c_1=0.$$

Furthermore

$$u(\varphi_0)=0,\quad \text{i.e.}\quad c_2=2a^2\int_0^{\varphi_0}\frac{\int_0^x(\sin\omega)^{n-1}d\omega}{(\sin x)^{n-1}}dx,$$

thus,

$$u(\varphi)=2a^2\int_{\varphi}^{\varphi_0}\frac{\int_0^x(\sin\omega)^{n-1}d\omega}{(\sin x)^{n-1}}dx.$$

■

Remark 3.1 *Notice that $u(\varphi)$ is an elementary function since the integral can be computed explicitly for every $n\geq 2$.*

Example 3.1 *Let $\varphi_0\in[0,\pi)$ be fixed. We consider the set D in S^2, such that*

$$D=\{(\theta,\varphi)|\,\theta\in[0,2\pi),\quad \text{and}\quad \varphi\in[0,\varphi_0)\}.$$

Of course,

$$\partial D=\{(\theta,\varphi)|\,\theta\in[0,2\pi)\quad \text{and}\quad \varphi=\varphi_0\}.$$

If X_t is the Brownian motion on S^2 starting at the point

$$A(\theta,\varphi)\in D,$$

then from (3.4) the expectation of T is given by

$$E^A[T]=2a^2\ln\left(\frac{1+\cos\varphi}{1+\cos\varphi_0}\right). \tag{3.10}$$

Example 3.2 *Let $\varphi_0\in[0,\pi)$ be fixed. We consider the set D in S^3, such that*

$$D=\{(\theta_1,\theta_2,\varphi)|\,\theta_1\in[0,2\pi),\theta_2\in[0,\pi]\quad \text{and}\quad \varphi\in[0,\varphi_0)\}.$$

Of course,

$$\partial D = \{(\theta_1, \theta_2, \varphi) | \, \theta_1 \in [0, 2\pi), \theta_2 \in [0, \pi] \quad \textit{and} \quad \varphi = \varphi_0\}\,.$$

If X_t is the Brownian motion on S^3 starting at the point

$$A = (\theta_1, \theta_2, \varphi) \in D,$$

then from (3.4) the expectation of T is given by

$$E^A[T] = a^2\left(\varphi \cot\varphi - \varphi_0 \cot\varphi_0\right). \tag{3.11}$$

Proposition 3.3 *Let $\varphi_1, \varphi_2 \in (0, \pi)$, such that $\varphi_1 < \varphi_2$, are both fixed. We consider the set D in $S^n, n \geq 2$, such that*

$$D = \{(\theta_1, \ldots, \theta_{n-1}, \varphi)|\,, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad \textit{for} \quad i = 2, \ldots, n-1 \quad \textit{and} \quad \varphi \in (\varphi_1, \varphi_2)\}\,.$$

Of course,

$$\partial D = \{(\theta_1, \ldots, \theta_{n-1}, \varphi)|\,, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad \textit{for} \quad i = 2, \ldots, n-1 \quad \textit{and}$$

$$\varphi = \varphi_1 \quad \textit{or} \quad \varphi = \varphi_2 \quad \}\,.$$

If X_t is the Brownian motion on S^n starting at the point

$$A = (\theta_1, \ldots, \theta_{n-1}, \varphi) \in D$$

then the expectation of T is given by

$$E^A[T] = 2a^2\left(\int_{\varphi}^{\varphi_1} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx + \frac{\int_{\varphi_1}^{\varphi_2} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx}{\int_{\varphi 1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx} \cdot \int_{\varphi_1}^{\varphi} \frac{1}{(\sin x)^{n-1}} dx\right). \tag{3.12}$$

Proof.

By the solution of the stochastic Poisson problem (see [7])

$$u(\theta_1, \ldots, \theta_{n-1}, \varphi) = E^A[T]$$

is the unique solution of the differential equation (3.5)
i.e.

$$\frac{1}{2}\Delta_n u = -1,$$

with boundary condition

$$u(\theta_1, \ldots, \theta_{n-1}, \varphi_1) = u(\theta_1, \ldots, \theta_{n-1}, \varphi_2) = 0.$$

Here Δ_n is the Laplace-Beltrami operator on S^n.
By the symmetry of D, it follows that the expectation value of T is independent of $\theta_1, \ldots, \theta_{n-1}$. Hence u is independent of θ_i, for $i = 1, \ldots, n-1$. The differential equation (3.5) takes the form (3.7) with boundary condition

$$u(\theta_1, \ldots, \theta_{n-1}, \varphi_1) = u(\theta_1, \ldots, \theta_{n-1}, \varphi_2) = 0. \tag{3.13}$$

Hence from (3.9)

$$u(\varphi) = -2a^2 \int_0^{\varphi} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx + c_1 \int_0^{\varphi} \frac{1}{(\sin x)^{n-1}} dx + c_2.$$

However

$$u(\theta_1, \ldots, \theta_{n-1}, \varphi_1) = u(\theta_1, \ldots, \theta_{n-1}, \varphi_2) = 0,$$

Thus

$$c_1 = 2a^2 \frac{\int_{\varphi_1}^{\varphi_2} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx}.$$

and

$$c_2 = 2a^2 \left(\int_0^{\varphi_1} \frac{\int_0^x (\sin x)^{n-1} d\omega}{(\sin x)^{n-1}} dx - \frac{\int_{\varphi_1}^{\varphi_2} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx} \cdot \int_0^{\varphi_1} \frac{1}{(\sin x)^{n-1}} dx \right).$$

Therefore

$$E^A[T] = 2a^2 \left(\int_{\varphi}^{\varphi_1} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx + \frac{\int_{\varphi_1}^{\varphi_2} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx} \cdot \int_{\varphi_1}^{\varphi} \frac{1}{(\sin x)^{n-1}} dx \right).$$

■

Example 3.3 *Let $\varphi_1, \varphi_2 \in (0, \pi)$, such that $\varphi_1 < \varphi_2$, are both fixed. We consider the set D in S^2, such that*

$$D = \{(\theta, \varphi) | \, \theta \in [0, 2\pi), \quad \text{and} \quad \varphi \in (\varphi_1, \varphi_2)\}.$$

Of course,

$$\partial D = \{(\theta, \varphi) | \, \theta \in [0, 2\pi) \quad \text{and} \quad \varphi = \varphi_1 \quad \text{or} \quad \varphi = \varphi_2 \quad \}.$$

If X_t is the Brownian motion on S^2 starting at the point

$$A = (\theta, \varphi) \in D,$$

then from (3.12) the expectation of T is given by

$$E^A[T] = \frac{4a^2}{\ln\left(\frac{\tan(\frac{\varphi_2}{2})}{\tan(\frac{\varphi_1}{2})}\right)} \left[\ln\left(\frac{\cos\left(\frac{\varphi_1}{2}\right)}{\cos\left(\frac{\varphi_2}{2}\right)}\right) \cdot \ln\left(\frac{\sin\left(\frac{\varphi}{2}\right)}{\sin\left(\frac{\varphi_1}{2}\right)}\right) - \ln\left(\frac{\cos\left(\frac{\varphi_1}{2}\right)}{\cos\left(\frac{\varphi}{2}\right)}\right) \cdot \ln\left(\frac{\sin\left(\frac{\varphi_2}{2}\right)}{\sin\left(\frac{\varphi_1}{2}\right)}\right) \right]. \tag{3.14}$$

Example 3.4 *Let $\varphi_1, \varphi_2 \in (0, \pi)$, such that $\varphi_1 < \varphi_2$, are both fixed. We consider the set D in S^3, such that*

$$D = \{(\theta_1, \theta_2, \varphi) | \, \theta_1 \in [0, 2\pi), \theta_2 \in [0, \pi], \quad \text{and} \quad \varphi \in (\varphi_1, \varphi_2)\}.$$

Of course,

$$\partial D = \{(\theta_1, \theta_2, \varphi) | \, \theta_1 \in [0, 2\pi), \theta_2 \in [0, \pi] \quad \text{and} \quad \varphi = \varphi_1 \quad \text{or} \quad \varphi = \varphi_2 \quad \}.$$

If X_t *is the Brownian motion on* S^3 *starting at the point*

$$A = (\theta_1, \theta_2, \varphi) \in D,$$

then from (3.12) the expectation of T *is given by*

$$E^A[T] = \frac{a^2\left[(\varphi-\varphi_1)\cot\varphi\cot\varphi_1 + (\varphi_1-\varphi_2)\cot\varphi_1\cot\varphi_2 + (\varphi_2-\varphi)\cot\varphi_2\cot\varphi\right]}{\cot\varphi_1 - \cot\varphi_2}. \tag{3.15}$$

3.2 Hitting probabilities

Proposition 3.4 *Let* $\varphi_1, \varphi_2 \in [0, 2\pi)$, *such that* $\varphi_1 < \varphi_2$, *are both fixed. We consider the sets* D_1, D_2 *in* S^1, *such that*

$$D_1 = (\varphi_1, 2\pi) \quad \text{and} \quad D_2 = [0, \varphi_2).$$

Of course,

$$\partial D_1 = \{\varphi_1\}. \quad \text{and} \quad \partial D_2 = \{\varphi_2\}.$$

Let X_t *is the Brownian motion on* S^1 *starting at the point*

$$A \in D_1 \cap D_2.$$

If

$$T_1 = \inf\{t \geq 0 \,|, X_t \notin D_1\}, T_2 = \inf\{t \geq 0 \,|, X_t \notin D_2\}$$

and

$$T = \inf\{t \geq 0 \,|, X_t \notin D_1 \cap D_2\},$$

then

$$Pr^A\{T = T_1\} = \frac{\varphi_2 - \varphi}{\varphi_2 - \varphi_1} \quad \text{and} \quad Pr^A\{T = T_2\} = \frac{\varphi - \varphi_1}{\varphi_2 - \varphi_1} \tag{3.16}$$

Proof.
It is known that, (see [6]), the function

$$u(\varphi) = Pr^A\{T = T_1\}$$

is the unique solution of the differential equation

$$\frac{1}{2}\Delta_1 u = 0 \tag{3.17}$$

with boundary condition

$$u(\varphi_1) = 1 \quad \text{and} \quad u(\varphi_2) = 0. \tag{3.18}$$

Here Δ_1 is the Laplace-Beltrami operator on S^1. Hence

$$u(\varphi) = \frac{\varphi_2 - \varphi}{\varphi_2 - \varphi_1}.$$

Therefore

$$Pr^A\{T = T_1\} = \frac{\varphi_2 - \varphi}{\varphi_2 - \varphi_1} \quad \text{and} \quad Pr^A\{T = T_2\} = \frac{\varphi - \varphi_1}{\varphi_2 - \varphi_1}.$$

∎

Proposition 3.5 *Let $\varphi_1, \varphi_2 \in (0, \pi)$, such that $\varphi_1 < \varphi_2$, are both fixed. We consider the sets D_1, D_2 in $S^n, n \geq 2$, such that*

$$D_1 = \{(\theta_1, \ldots, \theta_{n-1}, \varphi)|, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad for \quad i = 2, \ldots, n-1 \quad and \quad \varphi \in (\varphi_1, \pi]\}$$

and

$$D_2 = \{(\theta_1, \ldots, \theta_{n-1}, \varphi)|, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad for \quad i = 2, \ldots, n-1 \quad and \quad \varphi \in [0, \varphi_2)\}.$$

Of course,

$$\partial D_1 = \{(\theta_1, \ldots, \theta_{n-1}, \varphi)|, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad for \quad i = 2, \ldots, n-1 \quad and \quad \varphi = \varphi_1\}$$

and

$$\partial D_2 = \{(\theta_1, \ldots, \theta_{n-1}, \varphi)|, \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad for \quad i = 2, \ldots, n-1 \quad and \quad \varphi = \varphi_2\}.$$

Let X_t is the Brownian motion on S^n starting at the point

$$A = (\theta_1, \ldots, \theta_{n-1}, \varphi) \in D_1 \cap D_2.$$

If

$$T_1 = \inf\{t \geq 0 |, X_t \notin D_1\}, T_2 = \inf\{t \geq 0 |, X_t \notin D_2\}$$

and

$$T = \inf\{t \geq 0 |, X_t \notin D_1 \cap D_2\},$$

then

$$Pr^A\{T = T_1\} = \frac{\int_{\varphi}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx} \quad and \quad Pr^A\{T = T_2\} = \frac{\int_{\varphi_1}^{\varphi} \frac{1}{(\sin x)^{n-1}} dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx} \tag{3.19}$$

Proof.

It is known that (see [6]),

$$u(\theta_1, \ldots, \theta_{n-1}, \varphi) = Pr^A\{T = T_1\}$$

is the unique solution of the differential equation

$$\frac{1}{2}\Delta_n u = 0, \tag{3.20}$$

with boundary condition

$$u(\theta_1, \ldots, \theta_{n-1}, \varphi_1) = 1 \quad \text{and} \quad u(\theta_1, \ldots, \theta_{n-1}, \varphi_2) = 0$$

Here Δ_n is the Laplace-Beltrami operator on S^n.
By the symmetry of D, it follows that the probability of $T = T_1$ is independent of $\theta_1, \ldots, \theta_{n-1}$. Hence u is independent of θ_i, for $i = 1, \ldots, n-1$. From (1.2) the differential equation (3.20) takes the form

$$\frac{1}{2a^2}\left[(n-1)\cot(\varphi)\frac{du}{d\varphi} + \frac{d^2u}{d\varphi^2}\right] = 0, \tag{3.21}$$

with boundary condition

$$u(\varphi_1) = 1 \quad \text{and} \quad u(\varphi_2) = 0. \tag{3.22}$$

Set

$$f(\varphi) = \frac{du}{d\varphi},$$

hence from (3.21)

$$\frac{1}{2a^2}\left[(n-1)\cot(\varphi)f(\varphi) + \frac{df(\varphi)}{d\varphi}\right] = 0,$$

or

$$(n-1)\cos(\varphi)f(\varphi) + \sin(\varphi)\frac{df(\varphi)}{d\varphi} = 0.$$

Multiplying by $(\sin\varphi)^{n-2}$ implies that

$$(n-1)(\sin\varphi)^{n-2}\cos(\varphi)f(\varphi) + (\sin\varphi)^{n-1}\frac{df(\varphi)}{d\varphi} = 0,$$

or

$$\frac{d}{d\varphi}\left[(\sin\varphi)^{n-1}f(\varphi)\right] = 0.$$

Thus

$$f(\varphi) = \frac{c_1}{(\sin\varphi)^{n-1}},$$

i.e.

$$u(\varphi) = \int_0^{\varphi} \frac{c_1}{(\sin x)^{n-1}}dx + c_2. \tag{3.23}$$

However

$$u(\varphi_1) = 1 \quad \text{and} \quad u(\varphi_2) = 0,$$

hence

$$c_1 = -\frac{1}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}}dx} \quad \text{and} \quad c_2 = \frac{\int_0^{\varphi_2} \frac{1}{(\sin x)^{n-1}}dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}}dx}.$$

Therefore

$$Pr^A\{T = T_1\} = \frac{\int_{\varphi}^{\varphi_2} \frac{1}{(\sin x)^{n-1}}dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}}dx} \quad \text{and} \quad Pr^A\{T = T_2\} = \frac{\int_{\varphi_1}^{\varphi} \frac{1}{(\sin x)^{n-1}}dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}}dx}.$$

■

Example 3.5 *Let $\varphi_1, \varphi_2 \in (0,\pi)$, such that $\varphi_1 < \varphi_2$, are both fixed. We consider the sets D_1, D_2 in S^2, such that*

$$D_1 = \{(\theta,\varphi)|, \theta \in [0, 2\pi) \quad \text{and} \quad \varphi \in (\varphi_1, \pi]\}$$

and

$$D_2 = \{(\theta,\varphi)|, \theta \in [0, 2\pi) \quad \text{and} \quad \varphi \in [0, \varphi_2)\}.$$

Of course,

$$\partial D_1 = \{(\theta,\varphi)|\,,\theta \in [0,2\pi), \quad \mathit{and} \quad \varphi = \varphi_1\}$$

and

$$\partial D_2 = \{(\theta,\varphi)|\,,\theta_1 \in [0,2\pi), \quad \mathit{and} \quad \varphi = \varphi_2\}.$$

Let X_t is the Brownian motion on S^2 starting at the point

$$A = (\theta,\varphi) \in D_1 \cap D_2.$$

If

$$T_1 = \inf\{t \geq 0\,|, X_t \notin D_1\}, T_2 = \inf\{t \geq 0\,|, X_t \notin D_2\}$$

and

$$T = \inf\{t \geq 0\,|, X_t \notin D_1 \cap D_2\},$$

then from (3.19)

$$Pr^A\{T = T_1\} = \frac{\ln\left(\frac{\tan(\frac{\varphi_2}{2})}{\tan(\frac{\varphi}{2})}\right)}{\ln\left(\frac{\tan(\frac{\varphi_2}{2})}{\tan(\frac{\varphi_1}{2})}\right)} \quad \mathit{and} \quad Pr^A\{T = T_2\} = \frac{\ln\left(\frac{\tan(\frac{\varphi}{2})}{\tan(\frac{\varphi_1}{2})}\right)}{\ln\left(\frac{\tan(\frac{\varphi_2}{2})}{\tan(\frac{\varphi_1}{2})}\right)} \tag{3.24}$$

Example 3.6 *Let $\varphi_1, \varphi_2 \in (0,\pi)$, such that $\varphi_1 < \varphi_2$, both fixed. We consider the sets D_1, D_2 in S^3, such that*

$$D_1 = \{(\theta_1,\theta_2,\varphi)|\,,\theta_1 \in [0,2\pi), \theta_2 \in [0,\pi] \quad \mathit{and} \quad \varphi \in (\varphi_1,\pi]\}$$

and

$$D_2 = \{(\theta_1,\theta_2,\varphi)|\,,\theta_1 \in [0,2\pi), \theta_2 \in [0,\pi] \quad \mathit{and} \quad \varphi \in [0,\varphi_2)\}.$$

Of course,

$$\partial D_1 = \{(\theta_1,\theta_2,\varphi)|\,,\theta_1 \in [0,2\pi), \theta_2 \in [0,\pi] \quad \mathit{and} \quad \varphi = \varphi_1\}$$

and

$$\partial D_2 = \{(\theta_1,\theta 2,\varphi)|\,,\theta_1 \in [0,2\pi), \theta_2 \in [0,\pi] \quad \mathit{and} \quad \varphi = \varphi_2\}.$$

Let X_t is the Brownian motion on S^3 starting at the point

$$A = (\theta_1,\theta_2,\varphi) \in D_1 \cap D_2.$$

If

$$T_1 = \inf\{t \geq 0\,|, X_t \notin D_1\}, T_2 = \inf\{t \geq 0\,|, X_t \notin D_2\}$$

and

$$T = \inf\{t \geq 0\,|, X_t \notin D_1 \cap D_2\},$$

then from (3.19)

$$Pr^A\{T = T_1\} = \frac{\cot\varphi - \cot\varphi_2}{\cot\varphi_1 - \cot\varphi_2} \quad \mathit{and} \quad Pr^A\{T = T_2\} = \frac{\cot\varphi_1 - \cot\varphi}{\cot\varphi_1 - \cot\varphi_2} \tag{3.25}$$

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Asymptotic behavior of the solutions of a class of rational difference equations

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Abstract

In this paper we study the asymptotic behavior of the positive solutions of the rational difference equations

$$x_{n+1} = \frac{ax_{n-m(k+1)+1}}{\prod_{s=0}^{k} x_{n-m(s+1)+1} + 1}, \quad x_{n+1} = \frac{ax_{n-2k-1}\prod_{s=0}^{k} x_{n-2s}}{\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}},$$

$$x_{n+1} = \frac{ax_n x_{n-m(k+1)+1}}{x_n + x_{n-m(k+1)}}, \quad n = 0, 1, ...,$$

where $k, m \in \{1, 2, ...\}$ and a is a positive number.

Keywords: Difference equation, periodic solution, convergence of the solutions.

1 Introduction

In [8] the author studied the global behavior of the second order rational difference equation having quadratic term

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \quad a > 0, \; b > 0 \tag{1.1}$$

and the third order difference equation having quadratic term

$$x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \ a > 0, \ b > 0 \tag{1.2}$$

where x_{-2}, x_{-1}, x_0 are real numbers. For the study of equation (1.1) the author used the fact that (1.1) reduces to a linear nonhomogeneous equation. Moreover, for the study of (1.2) he showed that equation (1.2) reduces to (1.1).

Furthermore in [3] the authors investigated equation (1.1) with nonnegative initial values x_{-1}, x_0. Moreover if we get $b = 1$ in (1.1) then by dropping either the term x_n or x_{n-1} in the denominator of the equation (1.1) we obtain the equations

$$x_{n+1} = \frac{ax_{n-1}}{x_n + 1}, \quad x_{n+1} = \frac{ax_{n-1}}{x_{n-1} + 1}$$

which have been studied in [2]. Finally, results concerning rational difference equations having quadratic terms are included in [1], [3]-[11] and the references cited therein.

Now in this paper we study the following equations

$$x_{n+1} = \frac{ax_{n-m(k+1)+1}}{\prod_{s=0}^{k} x_{n-m(s+1)+1} + 1}, \ n = 0, 1, ... \tag{1.3}$$

$$x_{n+1} = \frac{ax_{n-2k-1} \prod_{s=0}^{k} x_{n-2s}}{\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}} \tag{1.4}$$

and

$$x_{n+1} = \frac{ax_n x_{n-m(k+1)+1}}{x_n + x_{n-m(k+1)}}, \quad n = 0, 1, ..., \tag{1.5}$$

where a is a positive number, $m, k \in \{1, 2, ...\}$ and the initial values of the above equations are positive numbers. More precisely, we study the existence of periodic solutions and the asymptotic behavior of the positive solutions for equations (1.3)-(1.5). We note that equations (1.3)-(1.5) have a common property: They reduces to a linear nonhomogeneous equation.

2 Study of equation (1.3)

First we study the existence of positive periodic solutions of period $m(k+1)$ for equation (1.3).

Proposition 2.1 *Consider equation* (1.3). *Suppose that*

$$a > 1. \tag{2.1}$$

Then equation (1.3) *has periodic solutions of period* $m(k+1)$.

Proof. Let x_n be a positive solution of (1.3) with initial values $x_{-m(k+1)+1}$, $x_{-m(k+1)+2}, ..., x_0$ are positive numbers such that

$$\prod_{s=0}^{k} x_{i-m(s+1)+1} = a - 1, \quad i = 0, 1, ..., m-1. \tag{2.2}$$

We prove that x_n is a periodic solution of (1.3) of period $m(k+1)$. From (1.3) and (2.2) we get

$$\begin{aligned}
x_1 &= \frac{ax_{-m(k+1)+1}}{\prod_{s=0}^{k} x_{-m(s+1)+1} + 1} = x_{-m(k+1)+1}, \\
x_2 &= \frac{ax_{-m(k+1)+2}}{\prod_{s=0}^{k} x_{-m(s+1)+2} + 1} = x_{-m(k+1)+2}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x_m &= \frac{ax_{-mk}}{\prod_{s=0}^{k} x_{-ms} + 1} = x_{-mk}.
\end{aligned} \tag{2.3}$$

Then from (1.3) and (2.3) we obtain

$$x_{m+1} = \frac{ax_{-mk+1}}{x_1 \prod_{s=1}^{k} x_{-ms+1} + 1} = \frac{ax_{-mk+1}}{x_{-m(k+1)+1} \prod_{s=1}^{k} x_{-ms+1} + 1} =$$

$$\frac{ax_{-mk+1}}{\prod_{s=0}^{k} x_{-m(s+1)+1} + 1} = x_{-mk+1}.$$

Working inductively we can prove that

$$x_{m+j} = x_{-mk+j}, \quad j = 2, 3, \ldots$$

and so the proof of the proposition is completed.

In the next proposition we study the asymptotic behavior of the positive solutions of (1.3). We need the following lemma.

Lemma 2.1 *Let x_n be an arbitrary positive solution of (1.3). Then the following statements are true:*

(i) If

$$t_n = \prod_{s=0}^{k} x_{n-sm}^{-1}, \quad n = 1, 2, \ldots \tag{2.4}$$

with

$$t_j = \prod_{s=0}^{k} x_{j-sm}^{-1}, \quad j = 1 - m, 2 - m, \ldots, 0, \tag{2.5}$$

then t_n satisfies the nonhomogeneous linear difference equation

$$y_{n+1} = \frac{1}{a} y_{n+1-m} + \frac{1}{a}, \quad n = 0, 1, \ldots \tag{2.6}$$

Moreover,

$$t_n = \begin{cases} B_n + \dfrac{n}{m}, \quad n = 1, 2, \ldots & \text{if } a = 1 \\ \\ \left(\dfrac{1}{a}\right)^{\frac{n}{m}} B_n + \dfrac{1}{a-1}, \quad n = 1, 2, \ldots & \text{if } a \neq 1 \end{cases} \tag{2.7}$$

where

$$B_n = \sum_{i=0}^{r} c_i \cos\left(\frac{2\pi n i}{m}\right) + d_i \sin\left(\frac{2\pi n i}{m}\right), \quad r = \begin{cases} \dfrac{m-1}{2}, & \text{if } m \text{ is odd} \\ \\ \dfrac{m}{2}, & \text{if } m \text{ is even} \end{cases} \tag{2.8}$$

and c_i, d_i, $i = 0, 1, \ldots r$ are constants which are derived from (2.5), (2.7) and (2.8).

(ii) If

$$y_n^{(j)} = x_{m(k+1)n+j}, \quad j = 0, 1, ..., m(k+1) - 1 \tag{2.9}$$

then

$$y_n^{(j)} = y_0^{(j)} \prod_{s=0}^{n-1} \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}, \quad j = 0, 1, ..., m(k+1) - 1. \tag{2.10}$$

Proof. Let x_n be an arbitrary solution of (1.3). Then we get

$$x_{n+1} \prod_{s=1}^{k} x_{n+1-sm} = \frac{a x_{n-m(k+1)+1} \prod_{s=1}^{k} x_{n+1-sm}}{\prod_{s=0}^{k} x_{n+1-(s+1)m} + 1}$$

which implies that

$$\prod_{s=0}^{k} x_{n+1-sm} = \frac{a \prod_{s=0}^{k} x_{n+1-(s+1)m}}{\prod_{s=0}^{k} x_{n+1-(s+1)m} + 1}. \tag{2.11}$$

Then from (2.4) and (2.11) we have

$$\frac{1}{t_{n+1}} = \frac{\frac{a}{t_{n+1-m}}}{\frac{1}{t_{n+1-m}} + 1}$$

which implies that t_n satisfies the difference equation (2.6). Then relations (2.7) and (2.8) follow immediately. This completes the proof of statement (i).

(ii) From (2.4) we have

$$\frac{t_n}{t_{n-m}} = \frac{x_n^{-1} x_{n-m}^{-1} \cdots x_{n-km}^{-1}}{x_{n-m}^{-1} x_{n-2m}^{-1} \cdots x_{n-(k+1)m}^{-1}} = \frac{x_{n-m(k+1)}}{x_n}$$

which implies that

$$x_n = \frac{t_{n-m}}{t_n} x_{n-m(k+1)}, \quad n = 1, 2, ... \tag{2.12}$$

So, from (2.9)) and (2.12) it holds

$$y_{n+1}^{(j)} = \frac{t_{m(k+1)(n+1)+j-m}}{t_{m(k+1)(n+1)+j}} y_n^{(j)}, \quad j = 0, 1, ..., m(k+1) - 1. \tag{2.13}$$

Therefore relation (2.13) implies that (2.10) is true. This completes the proof of the lemma.

Proposition 2.2 *Consider equation* (1.3). *Then the following statements are true.*

(i) If

$$0 < a \leq 1 \tag{2.14}$$

then every positive solution of (1.3) *tends to zero as* $n \to \infty$.

(ii) If (2.1) *holds then every positive solution of* (1.3) *tends to a periodic solution of period* $m(k+1)$.

Proof. Let x_n be an arbitrary positive solution of (1.3).

(i) Suppose first that

$$0 < a < 1. \tag{2.15}$$

From (1.3) we get for $j = 0, 1, ..., m(k+1) - 1$

$$x_{m(k+1)n+j} < a x_{m(k+1)(n-1)+j} < \cdots < a^n x_j. \tag{2.16}$$

Then from (2.15) and (2.16) we take

$$\lim_{n \to \infty} x_{m(k+1)n+j} = 0, \quad j = 0, 1, ..., m(k+1) - 1$$

which implies that x_n tends to zero as $n \to \infty$.

Let now $a = 1$. We consider the functions

$$A_n^{(j)} = \ln\left(\prod_{s=0}^{n-1} \frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right) = \sum_{s=0}^{n-1} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right). \tag{2.17}$$

From (2.8) it is obvious that

$$B_{m(k+1)(s+1)+j-m} = B_{m(k+1)(s+1)+j}, \quad s = 0, 1, ..., \quad j = 0, 1, ..., m(k+1) - 1. \tag{2.18}$$

Hence relations (2.7), (2.8) and (2.18) imply that

$$t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j} = -1, \quad s = 0, 1, ..., \quad j = 0, 1, ..., m(k+1) - 1. \tag{2.19}$$

In addition, if a is a real number such that $1+a>0$ then

$$ln(1+a) < a. \tag{2.20}$$

Then from (2.19) and (2.20) we get

$$\sum_{s=0}^{n-1} \ln\left(1 + \frac{t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}}{t_{m(k+1)(s+1)+j}}\right) \leq$$
$$\sum_{s=0}^{n-1}\left(\frac{t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}}{t_{m(k+1)(s+1)+j}}\right) = -\sum_{s=0}^{n-1}\frac{1}{t_{m(k+1)(s+1)+j}}. \tag{2.21}$$

Since from (2.7)

$$\sum_{s=0}^{\infty}\frac{1}{t_{m(k+1)(s+1)+j}} = \infty$$

then from (2.21)

$$\sum_{s=0}^{\infty} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right) = -\infty. \tag{2.22}$$

Therefore, from (2.17) and (2.22) we have

$$\lim_{n\to\infty} A_n^{(j)} = -\infty, \quad j = 0, 1, ..., m(k+1)-1$$

which implies that

$$\prod_{s=0}^{\infty}\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}} = 0, \quad j = 0, 1, ..., m(k+1)-1. \tag{2.23}$$

So from (2.10) and (2.23) we have that x_n tends to zero as $n \to \infty$. This completes the proof of statement (i).

(ii) If $a, b > 0$ then using (2.20) we can easily prove that

$$|\ln(\frac{a}{b})| \leq |a-b| \max\{\frac{1}{a}, \frac{1}{b}\}. \tag{2.24}$$

Then from (2.24) we have for $j = 0, 1, ..., m(k+1)-1$

$$\left|\sum_{s=0}^{n-1} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right)\right| \leq \sum_{s=0}^{n-1}\left|\ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right)\right| \leq$$
$$\sum_{s=0}^{n-1}|t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}| \max\left\{\frac{1}{t_{m(k+1)(s+1)+j}}, \frac{1}{t_{m(k+1)(s+1)+j-m}}\right\}. \tag{2.25}$$

Furthermore, from (2.1), (2.7) and (2.18) we have

$$|t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}| = \left(\frac{1}{a}\right)^{\frac{m(k+1)(s+1)+j}{m}} |B_{m(k+1)(s+1)+j}|(a-1). \tag{2.26}$$

Then using (2.7) and (2.26) we can prove that there exists a positive number $M > 0$ such that

$$|t_{m(k+1)(s+1)+j-m} - t_{m(k+1)(s+1)+j}| \max\left\{\frac{1}{t_{m(k+1)(s+1)+j}}, \frac{1}{t_{m(k+1)(s+1)+j-m}}\right\} \leq$$

$$M\left(\frac{1}{a}\right)^{\frac{m(k+1)(s+1)+j}{m}}, \quad j = 0, 1, ..., m(k+1) - 1. \tag{2.27}$$

Therefore, from (2.1), (2.25) and (2.27) it follows that

$$\left|\sum_{s=0}^{\infty} \ln\left(\frac{t_{m(k+1)(s+1)+j-m}}{t_{m(k+1)(s+1)+j}}\right)\right| < \infty. \tag{2.28}$$

Then using (2.17) and (2.28) it is obvious that there exist the

$$\lim_{n\to\infty} A_n^{(j)} = l_j < \infty, \quad j = 0, 1, ..., m(k+1) - 1. \tag{2.29}$$

Relations (2.9), (2.10), (2.17) and (2.29) imply that

$$\lim_{n\to\infty} x_{m(k+1)n+j} = p_j < \infty, \quad j = 0, 1, ..., m(k+1) - 1.$$

This completes the proof of the proposition.

3 Study of equation (1.4)

First we study the existence of positive solutions of period $2k+2$ for the equation (1.4).

Proposition 3.1 *Consider equation* (1.4) *where*

$$a > 2. \tag{3.1}$$

Then equation (1.4) *has positive periodic solutions of period* $2k+2$.

Proof. Let x_n be a positive solution of (1.4) with initial values such that

$$\prod_{s=0}^{k} x_{-2s} = \prod_{s=0}^{k} x_{-2s-1} = a-2. \tag{3.2}$$

Then from (1.4) and (3.2) we get

$$x_1 = \frac{ax_{-2k-1}\prod_{s=0}^{k} x_{-2s}}{\prod_{s=0}^{2k+1} x_{-s} + \prod_{s=0}^{k} x_{-2s} + \prod_{s=0}^{k} x_{-2s-1}} = \frac{a(a-2)x_{-2k-1}}{(a-2)^2+2(a-2)} = x_{-2k-1},$$

$$x_2 = \frac{ax_1x_{-2k}\prod_{s=1}^{k} x_{1-2s}}{x_1\prod_{s=1}^{2k+1} x_{1-s} + x_1\prod_{s=1}^{k} x_{1-2s} + \prod_{s=0}^{k} x_{-2s}} =$$

$$\frac{ax_{-2k-1}x_{-2k}\prod_{s=1}^{k} x_{1-2s}}{x_{-2k-1}\prod_{s=1}^{2k+1} x_{1-s} + x_{-2k-1}\prod_{s=1}^{k} x_{1-2s} + \prod_{s=0}^{k} x_{-2s}} =$$

$$\frac{ax_{-2k}\prod_{s=0}^{k} x_{-2s-1}}{\prod_{s=0}^{2k+1} x_{-s} + \prod_{s=0}^{k} x_{-2s-1} + \prod_{s=0}^{k} x_{-2s}} = \frac{a(a-2)x_{-2k}}{(a-2)^2+2(a-2)} = x_{-2k}.$$

Working inductively we can prove that

$$x_n = x_{n-2k-2}, \ n = 3, 4, \ldots$$

This completes the proof of the proposition.

In the following proposition we study the asymptotic behavior of the positive solutions of (1.4). We need the following lemma.

Lemma 3.1 *Let x_n be a positive solution of (1.4). Then the following statements are true:*

(i) If

$$t_n = \prod_{s=0}^{k} x_{n-2s}^{-1}, \quad n = 1, 2, ... \tag{3.3}$$

with

$$t_j = \prod_{s=0}^{k} x_{j-2s}^{-1}, \quad j = -1, 0, \tag{3.4}$$

then t_n, $n = 1, 2, ...$ satisfies the following difference equation

$$y_{n+1} = \frac{1}{a} y_n + \frac{1}{a} y_{n-1} + \frac{1}{a}, \quad n = 0, 1, ... \tag{3.5}$$

Moreover,

$$t_n = \begin{cases} c_1(-\frac{1}{2})^n + c_2 + \frac{1}{3}n, & n = 1, 2, ... \quad \text{if } a = 2 \\ \\ c_1 p_1^n + c_2 p_2^n + \frac{1}{a-2}, & n = 1, 2, ... \quad \text{if } a \neq 2 \end{cases} \tag{3.6}$$

where

$$p_1 = \frac{1}{2a}(1 - \sqrt{1+4a}), \quad p_2 = \frac{1}{2a}(1 + \sqrt{1+4a}), \tag{3.7}$$

c_1, c_2 are defined from (3.4) and (3.6).

(ii) If

$$y_n^{(j)} = x_{2(k+1)n+j}, \quad j = 0, 1, ..., 2k+1 \tag{3.8}$$

then

$$y_n^{(j)} = y_0^{(j)} \prod_{s=0}^{n-1} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}}, \quad j = 0, 1, ..., 2k+1. \tag{3.9}$$

Proof. (i) Let x_n be an arbitrary positive solution of (1.4). Then we get

$$x_{n+1} \prod_{s=1}^{k} x_{n-2s+1} = \frac{a x_{n-2k-1} \prod_{s=0}^{k} x_{n-2s} \prod_{s=1}^{k} x_{n-2s+1}}{\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}}$$

which implies that

$$\prod_{s=0}^{k} x_{n-2s+1} = \frac{a\prod_{s=0}^{2k+1} x_{n-s}}{\prod_{s=0}^{2k+1} x_{n-s} + \prod_{s=0}^{k} x_{n-2s} + \prod_{s=0}^{k} x_{n-2s-1}}. \tag{3.10}$$

Then relations (3.3) and (3.10) imply that

$$\frac{1}{t_{n+1}} = \frac{\frac{a}{t_n t_{n-1}}}{\frac{1}{t_n t_{n-1}} + \frac{1}{t_n} + \frac{1}{t_{n-1}}}$$

from which we take that t_n satisfies the difference equation (3.5). Then relation (3.6) follows immediately.

(ii) Using (3.3) we take

$$\frac{t_n}{t_{n-2}} = \frac{x_n^{-1}x_{n-2}^{-1}\cdots x_{n-2k}^{-1}}{x_{n-2}^{-1}x_{n-4}^{-1}\cdots x_{n-2k-2}^{-1}} = \frac{x_{n-2k-2}}{x_n}$$

which implies that

$$x_n = \frac{t_{n-2}}{t_n} x_{n-2k-2}, \quad n = 1, 2, ... \tag{3.11}$$

So, from (3.8) and (3.11) it holds

$$y_n^{(j)} = \frac{t_{2(k+1)n+j-2}}{t_{2(k+1)n+j}} y_{n-1}^{(j)}, \quad j = 0, 1, ..., 2k+1. \tag{3.12}$$

From (3.12) relation (3.9) follows immediately. This completes the proof of the lemma.

Proposition 3.2 *Consider equation* (1.4). *Then the following statements are true:*

(ii) If

$$0 < a \leq 2 \tag{3.13}$$

then every positive solution of (1.4) *tends to zero as* $n \to \infty$.

(ii) If

$$a > 2 \tag{3.14}$$

then every positive solution of (1.4) *tends to a periodic solution of* (1.4) *of period* $2k+2$.

Proof. Let x_n be an arbitrary positive solution of (1.4).

(i) Suppose that (2.15) is satisfied. Relation (1.4) implies that for $j = 0, 1, ..., 2k+1$

$$x_{2(k+1)n+j} < a x_{2(k+1)(n-1)+j} < ... < a^n x_j. \tag{3.15}$$

Therefore, from (2.15) and (3.15) we get

$$\lim_{n\to\infty} x_{2(k+1)n+j} = 0, \quad j = 0, 1, ..., 2k+1 \tag{3.16}$$

which imply that x_n tends to zero as $n \to \infty$.

Suppose that

$$1 \le a < 2. \tag{3.17}$$

From (3.7) and (3.17) we can easily prove that

$$|p_1| < 1, \quad 1 < p_2. \tag{3.18}$$

We set for $j = 0, 1, ..., 2k+1$

$$B_n^{(j)} = \ln\left(\prod_{s=0}^{n-1} \frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}}\right). \tag{3.19}$$

Then from (2.20) we have for $j = 0, 1, ..., 2k+1$

$$\begin{aligned} B_n^{(j)} &= \sum_{s=0}^{n-1} \ln\left(1 + \frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}\right) \le \\ &\sum_{s=0}^{n-1}\left(\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}\right). \end{aligned} \tag{3.20}$$

Moreover, from (3.6) and (3.20) we can prove that

$$\frac{t_{2(k+1)(s+1)+j-2} - t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}} = \frac{c_1(p_1^{-2}-1)\left(\frac{p_1}{p_2}\right)^{2(k+1)(s+1)+j} + c_2(p_2^{-2}-1)}{c_1\left(\frac{p_1}{p_2}\right)^{2(k+1)(s+1)+j} + c_2 + \frac{1}{a-2}p_2^{-2(k+1)(s+1)+j}}. \tag{3.21}$$

Using (3.18) and (3.21) we have that

$$\lim_{s\to\infty}\left(\frac{t_{2(k+1)(s+1)+j-2}-t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}\right)=p_2^{-2}-1<0. \tag{3.22}$$

Therefore, from (3.20) and (3.22) we can prove that

$$\lim_{n\to\infty} B_n^{(j)}=-\infty,\quad j=0,1,...,2k+1 \tag{3.23}$$

which from (3.19) imply that for $j=0,1,...,2k+1$

$$\prod_{s=0}^{\infty}\frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}}=0. \tag{3.24}$$

Hence, from (3.8), (3.9) and (3.24) we have that relations (3.16) are true and so x_n tends to zero as $n\to\infty$.

Suppose now that

$$a=2. \tag{3.25}$$

So from (3.6) and (3.25) we get

$$\frac{t_{2(k+1)(s+1)+j-2}-t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}=$$
$$\frac{3c_1\left(-\frac{1}{2}\right)^{2(k+1)(s+1)+j}-\frac{2}{3}}{c_2+c_1\left(-\frac{1}{2}\right)^{2(k+1)(s+1)+j}+\frac{1}{3}(2(k+1)(s+1)+j)}. \tag{3.26}$$

Then from (3.26) we can easily prove

$$\sum_{s=0}^{\infty}\left(\frac{t_{2(k+1)(s+1)+j-2}-t_{2(k+1)(s+1)+j}}{t_{2(k+1)(s+1)+j}}\right)=-\infty. \tag{3.27}$$

Therefore, from (3.20), (3.27) we have that (3.23) is satisfied and so arguing as above (3.16) hold which implies that x_n tends to zero as $n\to\infty$.

(ii) Finally, suppose that (3.14) is satisfied. Then from (3.7) it is obvious that

$$\left|\frac{p_1}{p_2}\right|<1,\ |p_1|<1,\ \ p_2<1. \tag{3.28}$$

In addition, from (3.6) we have that for $j=0,1,...,2k+1$

$$t_{2(k+1)(s+1)+j-2}-t_{2(k+1)(s+1)+j}=$$
$$p_2^{2(k+1)(s+1)+j}\left(c_1(p_1^{-2}-1)\left(\frac{p_1}{p_2}\right)^{2(k+1)(s+1)+j}+c_2(p_2^{-2}-1)\right). \tag{3.29}$$

In addition, from (2.24) we get for $j = 0, 1, ..., 2k+1$

$$\left| \ln \left(\frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} \right) \right| \leq$$

$$|t_{2(k+1)(s+1)-2+j} - t_{2(k+1)(s+1)+j}| \max \left\{ \frac{1}{t_{2(k+1)(s+1)-2+j}}, \frac{1}{t_{2(k+1)(s+1)+j}} \right\}. \tag{3.30}$$

Using (3.6), (3.28), (3.29) and (3.30) there exists a positive number N such that for $j = 0, 1, ..., 2k+1$

$$\left| \ln \left(\frac{t_{2(k+1)(s+1)+j-2}}{t_{2(k+1)(s+1)+j}} \right) \right| \leq N p_2^{2(k+1)(s+1)+j}. \tag{3.31}$$

Therefore, from (3.19) and (3.31) we have that there exist

$$\lim_{n \to \infty} B_n^{(j)} = \mu_j < \infty, \quad j = 0, 1, ..., 2k+1. \tag{3.32}$$

Hence, relations (3.8), (3.9), (3.19) and (3.32) imply that

$$\lim_{n \to \infty} x_{2(k+1)n+j} = q_j < \infty, \quad j = 0, 1, ..., 2k+1$$

and so the proof of the proposition is completed.

4 Study of equation (1.5)

In the first proposition we study the existence of positive periodic solutions of (1.5) of period $m(k+1)$.

Proposition 4.1 *Consider equation* (1.5) *where* (3.25) *holds. Let* x_n *be positive solution of* (1.5) *such that*

$$x_0 = x_{-m(k+1)}. \tag{4.1}$$

Then x_n *is a periodic solution of* (1.5) *with period* $m(k+1)$.

Proof Let x_n be a positive solution of (1.5) such that (4.1) holds. Then from (1.5), (3.25) we get

$$x_1 = \frac{2x_0 x_{-m(k+1)+1}}{x_0 + x_{-m(k+1)}} = \frac{2x_0 x_{-m(k+1)+1}}{2x_0} = x_{-m(k+1)+1}$$

and working inductively we can prove that

$$x_n = x_{n-m(k+1)}, \quad n = 1, 2, ...$$

This completes the proof of the proposition.

In the last proposition of this paper we study the asymptotic behavior of the positive solutions of (1.5). We need the following lemma.

Lemma 4.1 *Consider equation* (1.5). *Let x_n be a positive solution of* (1.5). *Then if $a \neq 1$, for $j = 0, 1, ..., m(k+1) - 1$ and $n = 0, 1, ...$ it holds*

$$x_{nm(k+1)+j} = (a-1)^n x_j \prod_{s=1}^{n} \frac{1}{c(a-1)(\frac{1}{a})^{sm(k+1)+j} + 1} \tag{4.2}$$

where

$$c = \frac{x_{-m(k+1)}}{x_0} - \frac{1}{a-1}$$

and if $a = 1$, for $j = 0, 1, ..., m(k+1) - 1$ and $n = 0, 1, ...$ it holds

$$x_{nm(k+1)+j} = x_j \prod_{s=1}^{n} \frac{1}{d + sm(k+1) + j}, \quad d = \frac{x_{-m(k+1)}}{x_0}. \tag{4.3}$$

Proof. We set

$$y_n = \frac{x_{n-m(k+1)}}{x_n}. \tag{4.4}$$

Then from (1.5) and (4.4) we get

$$y_{n+1} = \frac{1}{a} y_n + \frac{1}{a}, \quad n = 0, 1, ... \tag{4.5}$$

So from (4.4) and (4.5) relations (4.2) and (4.3) follow immediately. This completes the proof of the lemma.

Proposition 4.2 *Consider equation* (1.5). *Then the following statements are true:*

(i) If $0 < a < 2$ then every positive solution of (1.5) *tends to zero as $n \to \infty$.*

(ii) If $a = 2$ then every positive solution of (1.5) *tends to a periodic solution of* (1.5) *of period $m(k+1)$ as $n \to \infty$.*

(iii) If $a > 2$ then every positive solution of (1.5) *tends to ∞ as $n \to \infty$.*

Proof. Let x_n be an arbitrary solution of (1.5).

(i) Suppose that (2.15) holds. Then using (1.5) and arguing as in Proposition 2.2 we can prove that x_n tends to zero as $n \to \infty$.

Suppose that

$$1 < a < 2. \tag{4.6}$$

Let for $j = 0, 1, ..., m(k+1) - 1$

$$D_n^{(j)} = \prod_{s=1}^{n} \frac{1}{c(a-1)(\frac{1}{a})^{sm(k+1)+j} + 1}. \tag{4.7}$$

We have for $j = 0, 1, ..., m(k+1) - 1$

$$\ln(D_n^{(j)}) = -\sum_{s=1}^{n} \ln\Big(c(a-1)(\frac{1}{a})^{sm(k+1)+j} + 1\Big). \tag{4.8}$$

In addition, from (2.20) we take

$$|\ln(1+a)| \leq \max\{a, \frac{-a}{1+a}\}. \tag{4.9}$$

Using (4.8) and (4.9) and since

$$1 < a \tag{4.10}$$

we can prove that

$$\lim_{n\to\infty} (\ln(D_n^{(j)}) = L_j < \infty, \quad j = 0, 1, ..., m(k+1) - 1 \tag{4.11}$$

which implies that

$$\lim_{n\to\infty} D_n^{(j)} = M_j < \infty, \quad j = 0, 1, ..., m(k+1) - 1. \tag{4.12}$$

Therefore, from (4.2), (4.6), (4.7) and (4.12) we have that

$$\lim_{n\to\infty} x_{nm(k+1)+j} = 0, \quad j = 0, 1, ..., m(k+1) - 1 \tag{4.13}$$

and so x_n tends to zero as $n \to \infty$.

Let now $a = 1$. We set for $j = 0, 1, ..., m(k+1) - 1$

$$K_n^{(j)} = \prod_{s=1}^{n} \frac{1}{d + sm(k+1) + j}. \tag{4.14}$$

Then from (4.14) for $j = 0, 1, ..., m(k+1) - 1$ we take

$$\ln(K_n^{(j)}) = -\sum_{s=1}^{n} \ln\Big(d + sm(k+1) + j\Big). \tag{4.15}$$

So from (4.15) we can prove that

$$\lim_{n\to\infty} (\ln(K_n^{(j)}) = -\infty, \quad j = 0, 1, ..., m(k+1) - 1$$

which implies that

$$\lim_{n\to\infty} K_n^{(j)} = 0, \quad j = 0, 1, ..., m(k+1) - 1. \tag{4.16}$$

Then relations (4.3), (4.14), (4.16) imply that (4.13) are true and so x_n tends to zero as $n \to \infty$.

(ii) Suppose now that $a = 2$. Then from (4.10) relations (4.12) are true. So from (4.2) we have

$$\lim_{n\to\infty} x_{nm(k+1)+j} = M_j x_j < \infty, \quad j = 0, 1, ..., m(k+1) - 1$$

and so x_n tends to a periodic solution of (1.5) of period $m(k+1)$ as $n \to \infty$.

(iii) Finally, suppose that $a > 2$. Then using (4.10), we have that relations (4.12) hold and so from (4.2) it is clear that x_n tends to ∞ as $n \to \infty$. This completes the proof of the proposition.

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A COUNTEREXAMPLE TO THE NON-SEPARABLE VERSION OF ROSENTHAL'S ℓ_1-THEOREM

COSTAS POULIOS

The following theorem is due to H. P. Rosenthal [6] and it provides a fundamental criterion for the embedding of ℓ_1 in Banach spaces.

Theorem 1 (Rosenthal's ℓ_1-theorem). *Let (x_n) be a bounded sequence in the Banach space X and suppose that (x_n) has no weakly Cauchy subsequence. Then (x_n) must contain a subsequence which is equivalent to the usual ℓ_1-basis.*

First of all, we recall that the sequence (x_n) is called weakly Cauchy if for each continuous functional $f \in X^*$, the scalar sequence (fx_n) is Cauchy. We also say that the sequence (x_n) is equivalent to the usual ℓ_1-basis if there are constants $A,\ B > 0$ such that for any $n \in \mathbb{N}$ and any scalars $a_1, a_2, \ldots, a_n$,

$$A \sum_{i=1}^{n} |a_i| \le \| \sum_{i=1}^{n} a_i x_i \| \le B \sum_{i=1}^{n} |a_i|.$$

The above condition guarantees that the linear map $T : \ell_1 \to \overline{\text{span}}\{x_n \mid n \in \mathbb{N}\}$, defined by $Te_n = x_n$ for any $n \in \mathbb{N}$, is an isomorphism and therefore the space ℓ_1 embeds in X.

A satisfactory extension of Theorem 1 to spaces of the type $\ell_1(\kappa)$, for κ an uncountable cardinal, would be desirable, since it would provide a useful criterion for the embedding of $\ell_1(\kappa)$ in Banach spaces. Consequently, R. Haydon [4] posed the following problem: Let κ be an uncountable cardinal. Suppose that X is a Banach space and A is a bounded subset of X with $\text{card}(A) = \kappa$, such that A does not contain any weakly Cauchy sequence. Can we deduce that A has a subset equivalent to the usual basis of $\ell_1(\kappa)$?

Before posing the question, Haydon [3] exhibited a counterexample for the case where the cardinal κ is equal to ω_1. A completely different counterexample, for the case of ω_1, was also obtained by J. Hagler [2]. Finally, a complete solution to this problem was given by C. Gryllakis [1] who proved that the answer is always negative with only one exception, namely when κ and $\text{cf}(\kappa)$ are both strong limit cardinals. However, Gryllakis' proof is quite difficult and, unlike the case of ω_1, does not construct any specific counterexample.

In what follows, our aim is to present a counterexample to the non-separable version of Rosenthal's ℓ_1-theorem and to give a complete answer to Haydon's problem. More precisely, for any uncountable cardinal κ, we construct a non-separable analogue of the Hagler Tree space (see [2]). In the case where either κ or $\text{cf}(\kappa)$ is not a strong limit cardinal, using the aforementioned construction, we obtain a Banach space X and a bounded subset A of X with $\text{card}(A) = \kappa$ such that (1) A contains no weakly Cauchy sequence and (2) no subset of A is equivalent to the usual $\ell_1(\kappa)$-basis. In the case where κ and $\text{cf}(\kappa)$ are both strong limit cardinals, the answer to Haydon's problem is positive (see [1]).

In the following we fix an infinite cardinal κ and we set

$$\begin{aligned}\{0,1\}^{\kappa} &= \Big\{a : \{\xi < \kappa\} \to \{0,1\}\Big\}\\ &= \Big\{(a_\xi)_{\xi<\kappa} \mid a_\xi = 0 \text{ or } 1\Big\}\\ \mathcal{D} = \{0,1\}^{<\kappa} &= \bigcup\Big\{\{0,1\}^{\eta} \mid \mathrm{Ord}(\eta), \eta < \kappa\Big\}\\ &= \Big\{(a_\xi)_{\xi<\eta} \mid \eta \text{ is an ordinal number, } \eta < \kappa,\ a_\xi = 0 \text{ or } 1\Big\}.\end{aligned}$$

The set $\mathcal{D}$ is called the *standard tree* and its elements are called *nodes*. The elemens of the set $\{0,1\}^{\kappa}$ are called *branches*.

If s is a node and $s \in \{0,1\}^{\eta}$ we say that s is on the *η-th level of $\mathcal{D}$* and we denote the level of s by $\mathrm{lev}(s)$. The initial segment partial ordering, denoted by $\leq$, is defined as follows: if $s = (a_\xi)_{\xi<\eta_1}$ and $s' = (b_\xi)_{\xi<\eta_2}$ belong to $\mathcal{D}$ then $s \leq s'$ if and only if $\eta_1 \leq \eta_2$ and $a_\xi = b_\xi$ for any $\xi < \eta_1$.

A linearly ordered subset $\mathcal{I}$ of $\mathcal{D}$ is called a *segment* if for every $s < t < s'$, t belongs to $\mathcal{I}$ provided that s, s' belong to $\mathcal{I}$. Consider now a non-empty segment $\mathcal{I}$ and let η_1 be the least ordinal number such that there is a node s with $\mathrm{lev}(s) = \eta_1$ and $s \in \mathcal{I}$. Moreover, suppose that there are an ordinal number η and a node s' on the η-th level such that $s \leq s'$ for any $s \in \mathcal{I}$. Let η_2 be the least ordinal satisfying this property. Then we say that $\mathcal{I}$ is an *η_1-η_2 segment.*

A finite family $\{\mathcal{I}_j\}_{j=1}^{r}$ of segments is called *admissible* if the following properties are satisfied

(1) there exist ordinals $\eta_1 < \eta_2$ such that each $\mathcal{I}_j$ is an η_1-η_2 segment,
(2) $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ provided that $i \neq j$.

We next consider the vector space $c_{00}(\mathcal{D})$ of finitely supported functions $x : \mathcal{D} \to \mathbb{R}$. For a segment $\mathcal{I}$ of $\mathcal{D}$, we set $\mathcal{I}^*(x) = \sum_{s\in\mathcal{I}} x(s)$. Then, for any $x \in c_{00}(\mathcal{D})$ we define the norm

$$\|x\| = \sup\Big[\sum_{j=1}^{r} |\mathcal{I}_j^*(x)|^2\Big]^{1/2}$$

where the supremum is taken over all admissible families $\{\mathcal{I}_j\}_{j=1}^{r}$ of segments. We set X_κ the completion of $c_{00}(\mathcal{D})$ under this norm.

Now let $B = (a_\xi)_{\xi<\kappa}$ be any branch. Then B can be naturally identified with a maximal segment of $\mathcal{D}$, namely

$$B = \{s_0 < s_1 < \ldots < s_\eta < \ldots\}$$

where $s_0 = \emptyset$ and $s_\eta = (a_\xi)_{\xi<\eta}$. For any function $x \in c_{00}(\mathcal{D})$ we have already defined $B^*(x) = \sum_{s\in B} x(s)$. Clearly, $B^* : c_{00}(\mathcal{D}) \to \mathbb{R}$ is a linear functional of norm 1. This functional can be extended to a bounded functional on X_κ, which is denoted again by B^*. Let Γ be the set which contains the functionals B^* defined above. Clearly, Γ is a bounded subset of X_κ^* with $\mathrm{card}(\Gamma) = 2^\kappa$.

Concerning the space X_κ and the family of functionals Γ, we prove the following theorems.

Theorem 2. *Any sequence $(B_n^*)_{n\in\mathbb{N}}$ in Γ has a subsequence equivalent to the usual ℓ_1-basis. Therefore, Γ contains no weakly Cauchy sequence.*

Theorem 3. *No subset of Γ is equivalent to the usual basis of $\ell_1(\kappa^+)$.*

Now let κ be a cardinal number, which is not strong limit. This means that there exists cardinal $\lambda < \kappa$ such that $\kappa \leq 2^\lambda$. Consider the space X_λ and the corresponding family $\Gamma \subset X_\lambda^*$. Then we have $\mathrm{card}(\Gamma) = 2^\lambda$ and hence we can choose a subset A of Γ with $\mathrm{card}(A) = \kappa$. By Theorem 2, the set A contains no weakly Cauchy sequence. Furthermore, by Theorem 3, no subset of A is equivalent to the usual $\ell_1(\kappa)$-basis.

Moreover, in the case where κ is strong limit and $\mathrm{cf}(\kappa)$ is not a strong limit cardinal, using our construction, we obtain a Banach space X and a subset A of X with the desired properties.

Finally, the main properties of the spaces Hagler Tree [2] and James Tree [5], by which our construction is inspired, suggest the following conjecture for the spaces X_κ.

Conjecture. *The space X_κ does not contain a subspace isomorphic to $\ell_1(\kappa)$.*

Concerning the above conjecture, a partial result can be proved rather easily. For any node $s \in \mathcal{D}$, let $e_s \in X_\kappa$ be defined by $e_s(t) = 1$ if $t = s$ and $e_s(t) = 0$ otherwise. Now consider any branch B and the subspace $\overline{\mathrm{span}}\{e_s \mid s \in B\}$. Then this subspace contains no isomorphic copy of $\ell_1(\kappa)$.

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On the recursive sequence $x_{n+1} = A + \frac{x_{n-1}^p}{x_n^q}$

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Abstract

This paper studies the behavior of positive solutions of the difference equation

$$x_{n+1} = A + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, ...,$$

where $A, p, q \in (0, \infty)$ and $x_{-1}, x_0 \in (0, \infty)$.

Keywords: Difference equation, boundedness, persistence, attractivity, asymptotic stability, periodicity.

1 Introduction

Difference equations have been applied in several mathematical models in biology, economics, genetics, population dynamics etc. For this reason, there exists an increasing interest in studying difference equations (see [1]-[28] and the references cited therein).

The investigation of positive solutions of the following equation

$$x_n = A + \frac{x_{n-k}^p}{x_{n-m}^q}, \quad n = 0, 1, ...,$$

where $A, p, q \in [0, \infty)$ and $k, m \in N$, $k \neq m$, was proposed by Stević at numerous conferences. For some results in the area see, for example, [3], [4], [5], [8], [11], [12], [19], [22], [24], [25], [28].

In [22] the author studied the boundedness, the global attractivity, the oscillatory behavior and the periodicity of the positive solutions of the equation

$$x_{n+1} = a + \frac{x_{n-1}^p}{x_n^p}, \quad n = 0, 1, \dots ,$$

where a, p are positive constants and the initial conditions x_{-1}, x_0 are positive numbers (see also [5] for more results on this equation).

In [11] the authors obtained boundedness, persistence, global attractivity and periodicity results for the positive solutions of the difference equation

$$x_{n+1} = a + \frac{x_{n-1}}{x_n^p}, \quad n = 0, 1, \dots ,$$

where a, p are positive constants and the initial conditions x_{-1}, x_0 are positive numbers.

Motivating by the above papers, we study now the boundedness, the persistence, the existence of unbounded solutions, the attractivity, the stability of the positive solutions and the period two solutions of the difference equation

$$x_{n+1} = A + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \dots , \tag{1.1}$$

where A, p, q are positive constants and the initial values x_{-1}, x_0 are positive real numbers.

Finally equations, closely related to Eq. (1.1), are considered in [1]-[11], [14], [16]-[23], [26], [27], and the references cited therein.

2 Boundedness and persistence

The following result is essentially proved in [22]. Hence, we omit its proof.

Proposition 2.1 *If*

$$0 < p < 1, \tag{2.1}$$

then every positive solution of Eq.(1.1) is bounded and persists.

In the next proposition we obtain sufficient conditions for the existence of unbounded solutions of Eq.(1.1).

Proposition 2.2 *If*

$$p > 1 \tag{2.2}$$

then there exist unbounded solutions of Eq.(1.1).

Proof Let x_n be a solution of (1.1) with initial values x_{-1}, x_0 such that

$$x_{-1} > \max\left\{(A+1)^{\frac{p}{q}},\ \ (A+1)^{\frac{q}{p-1}}\right\},\quad x_0 < A+1. \tag{2.3}$$

Then from (1.1), (2.2), (2.3) we have,

$$\begin{aligned} x_1 &= A + \frac{x_{-1}^p}{x_0^q} > A + \frac{x_{-1}^p}{(A+1)^q} - x_{-1} + x_{-1} \\ &= A + x_{-1}\Big(\frac{x_{-1}^{p-1}}{(A+1)^q} - 1\Big) + x_{-1} > A + x_{-1}. \end{aligned} \tag{2.4}$$

$$x_2 = A + \frac{x_0^p}{x_1^q} < A + \frac{(A+1)^p}{x_{-1}^q} < A+1. \tag{2.5}$$

Moreover from (1.1), (2.3) we have

$$x_1 = A + \frac{x_{-1}^p}{x_0^q} > A + \frac{(A+1)^{\frac{qp}{p-1}}}{(A+1)^q} = A + (A+1)^{\frac{q}{p-1}} > (A+1)^{\frac{q}{p-1}}. \tag{2.6}$$

Then using (1.1), (2.3)-(2.6) and arguing as above we get

$$x_3 = A + \frac{x_1^p}{x_2^q} > A + \frac{x_1^p}{(A+1)^q} - x_1 + x_1 > A + x_1.$$

$$x_4 = A + \frac{x_2^p}{x_3^q} < A + \frac{(A+1)^p}{x_{-1}^q} < A+1.$$

Therefore working inductively we can prove that for $n = 0, 1, \ldots$

$$x_{2n+1} > A + x_{2n-1},\quad x_{2n} < A+1$$

which implies that

$$\lim_{n\to\infty} x_{2n+1} = \infty.$$

So x_n is unbounded. This completes the proof of the proposition.

3 Attractivity and Stability

In the following proposition we prove the existence of a positive equilibrium.

Proposition 3.1 *If either*

$$0 < q < p < 1 \tag{3.1}$$

or

$$0 < p < q \tag{3.2}$$

hold then Eq.(1.1) has a unique positive equilibrium $\bar{x}$.

Proof. A point $\bar{x} \in I\!R$ will be an equilibrium of Eq.(1.1) if and only if satisfies the following equation

$$F(x) = x^{p-q} - x + A = 0.$$

Suppose that (3.1) is satisfied. Since (3.1) holds and

$$F'(x) = (p-q)x^{p-q-1} - 1 \tag{3.3}$$

we have that F is increasing in $[0, (p-q)^{\frac{1}{-p+q+1}}]$ and F is decreasing in $[(p-q)^{\frac{1}{-p+q+1}}, \infty)$. Moreover $F(0) = A > 0$ and

$$\lim_{x \to \infty} F(x) = -\infty. \tag{3.4}$$

So if (3.1) holds we get that Eq.(1.1) has a unique equilibrium $\bar{x}$ in $(0, \infty)$.

Suppose now that (3.2) holds. We observe that $F(1) = A > 0$ and since from (3.2), (3.3) $F'(x) < 0$, we have that F is decreasing in $(0, \infty)$. Thus from (3.4) we obtain that Eq.(1.1) has a unique equilibrium $\bar{x}$ in $(0, \infty)$. The proof is complete.

In the sequel, we study the global asymptotic stability of the positive solutions of Eq.(1.1).

Proposition 3.2 *Consider Eq.(1.1). Suppose that either*

$$0 < p < 1 < q, \quad A > (p+q-1)^{\frac{1}{q-p+1}} \tag{3.5}$$

or (3.1) and

$$0 < p + q \leq 1. \tag{3.6}$$

hold. Then the unique positive equilibrium of Eq.(1.1) is globally asymptotically stable.

Proof. First we prove that every positive solution of Eq.(1.1) tends to the unique positive equilibrium $\bar{x}$ of Eq.(1.1).

Assume first that (3.5) are satisfied. Let x_n be a positive solution of Eq.(1.1). From (3.5) and Proposition 2.1 we have

$$0 < l = \liminf_{n\to\infty} x_n, \quad L = \limsup_{n\to\infty} x_n < \infty. \tag{3.7}$$

Then from (1.1) and (3.7) we get,

$$L \leq A + \frac{L^p}{l^q}, \quad l \geq A + \frac{l^p}{L^q}$$

and so

$$Ll^q \leq Al^q + L^p, \quad lL^q \geq AL^q + l^p.$$

Thus,

$$AL^q l^{q-1} + l^p l^{q-1} \leq Al^q L^{q-1} + L^p L^{q-1}.$$

This implies that

$$AL^{q-1} l^{q-1}(L - l) \leq L^{p+q-1} - l^{p+q-1}. \tag{3.8}$$

Suppose for a while that $p + q - 2 \geq 0$. We shall prove that $l = L$. Suppose on the contrary that $l < L$. If we consider the function x^{p+q-1} then there exists a $c \in (l, L)$ such that

$$\frac{L^{p+q-1} - l^{p+q-1}}{L - l} = (p + q - 1)c^{p+q-2} \leq (p + q - 1)L^{p+q-2}. \tag{3.9}$$

Then from (3.8) and (3.9) we obtain

$$AL^{q-1} l^{q-1} \leq (p + q - 1)L^{p+q-2}$$

or

$$AL^{1-p} l^{q-1} \leq p + q - 1. \tag{3.10}$$

Moreover, since from (1.1),

$$L \geq A, \quad l \geq A$$

from (3.5) and (3.10) we get

$$AA^{1-p} A^{q-1} = A^{q-p+1} \leq p + q - 1$$

which contradicts to (3.5). So $l = L$ which implies that x_n tends to the unique positive equilibrium $\bar{x}$.

Suppose that $p+q-2<0$. Then from (3.8) and arguing as above we get

$$AL^{q-1}l^{q-1} \leq (p+q-1)l^{p+q-2}.$$

Then arguing as above we can prove that x_n tends to the unique positive equilibrium $\bar{x}$.

Assume now that (3.6) holds. From (3.6) and (3.8) we obtain

$$AL^{q-1}l^{q-1}(L-l) \leq \frac{1}{L^{1-p-q}} - \frac{1}{l^{1-p-q}} = \frac{l^{1-p-q} - L^{1-p-q}}{L^{1-p-q}l^{1-p-q}} \leq 0$$

which implies that $L = l$. So every positive solution x_n of Eq.(1.1) tends to the unique positive equilibrium $\bar{x}$ of Eq.(1.1).

It remains to prove now that the unique positive equilibrium of Eq.(1.1) is locally asymptotically stable. The linearized equation about the positive equilibrium $\bar{x}$ is the following

$$y_{n+2} + q\bar{x}^{p-q-1}y_{n+1} - p\bar{x}^{p-q-1}y_n = 0. \tag{3.11}$$

Using Theorem 1.3.4. of [13] the linear equation (3.11) is asymptotically stable if and only if

$$q\bar{x}^{p-q-1} < -p\bar{x}^{p-q-1} + 1 < 2. \tag{3.12}$$

First assume that (3.5) hold. Since (3.5) hold then we obtain that

$$A > (p+q)^{\frac{p-q}{q+1-p}}(q+p-1). \tag{3.13}$$

From (3.5) and (3.13) we can easily prove that

$$F(c) > 0, \ \textit{where} \ c = (p+q)^{\frac{1}{q+1-p}}. \tag{3.14}$$

Therefore

$$\bar{x} > (p+q)^{\frac{1}{q+1-p}} \tag{3.15}$$

which implies that (3.12) is true. So in this case the unique positive equilibrium $\bar{x}$ of Eq.(1.1) is locally asymptotically stable.

Finally suppose that (3.1) and (3.6) are satisfied. Then we can prove that (3.14) is satisfied and so the unique positive equilibrium $\bar{x}$ of Eq.(1.1) satisfies (3.15). Therefore (3.12) are hold. This implies that the unique positive equilibrium $\bar{x}$ of Eq.(1.1) is locally asymptotically stable. This completes the proof of the proposition.

4 Study of 2-periodic solutions

Motivated by Lemma 1 of [5], in this section we show that there is a prime two periodic solution. Moreover we find solutions of (1.1) which converge to a prime two periodic solution.

Proposition 4.1 *Consider Eq.(1.1) where*

$$0 < p < 1 < q. \tag{4.1}$$

Assume that there exists a sufficient small positive real number ϵ_1, *such that*

$$\frac{1}{(A+\epsilon_1)^{q-p}} > \epsilon_1 \tag{4.2}$$

and

$$(A+\epsilon_1)^{\frac{p}{q}}\epsilon_1^{-\frac{1}{q}} < A + \epsilon_1^{-p/q}(A+\epsilon_1)^{\frac{p^2-q^2}{q}}. \tag{4.3}$$

Then Eq.(1.1) has a periodic solution of prime period two.

Proof. Let x_n be a positive solution of Eq.(1.1). It is obvious that if

$$x_{-1} = A + \frac{x_{-1}^p}{x_0^q}, \quad x_0 = A + \frac{x_0^p}{x_{-1}^q},$$

then x_n is periodic of period two. Consider the system

$$x = A + \frac{x^p}{y^q}, \quad y = A + \frac{y^p}{x^q}, \tag{4.4}$$

Then system (4.4) is equivalent to

$$y - A - \frac{y^p}{x^q} = 0, \quad y = \frac{x^{\frac{p}{q}}}{(x-A)^{\frac{1}{q}}} \tag{4.5}$$

and so we get the equation

$$G(x) = \frac{x^{\frac{p}{q}}}{(x-A)^{\frac{1}{q}}} - A - \frac{x^{\frac{p^2-q^2}{q}}}{(x-A)^{\frac{p}{q}}} = 0. \tag{4.6}$$

We obtain

$$G(x) = \frac{1}{(x-A)^{\frac{1}{q}}}\left(x^{\frac{p}{q}} - x^{\frac{p^2-q^2}{q}}(x-A)^{\frac{1-p}{q}}\right) - A$$

and so from (4.1)

$$\lim_{x \to A+} G(x) = \infty.$$

Moreover from (4.3) we can show that

$$G(A + \epsilon_1) < 0. \tag{4.7}$$

Therefore the equation $G(x) = 0$ has a solution $\bar{x} = A + \epsilon_0$, where $0 < \epsilon_0 < \epsilon_1$, in the interval $(A, A + \epsilon_1)$. We have

$$\bar{y} = \frac{\bar{x}^{\frac{p}{q}}}{(\bar{x} - A)^{\frac{1}{q}}}.$$

We consider the function

$$H(\epsilon) = (A + \epsilon)^{p-q} - \epsilon.$$

Since form (4.1) $H'(\epsilon) = (p - q)(A + \epsilon)^{p-q-1} - 1 < 0$ we have

$$H(\epsilon_0) > H(\epsilon_1). \tag{4.8}$$

From (4.2) we have $H(\epsilon_1) > 0$, so from (4.8)

$$H(\epsilon_0) = (A + \epsilon_0)^{p-q} - \epsilon_0 > 0$$

which implies that

$$\bar{x} = A + \epsilon_0 < \frac{(A + \epsilon_0)^{\frac{p}{q}}}{\epsilon_0^{\frac{1}{q}}} = \bar{y}.$$

Hence, if $x_{-1} = \bar{x}$, $x_0 = \bar{y}$, then the solution x_n with initial values x_{-1}, x_0 is a prime 2-periodic solution.

In the sequel, we shall need the following lemmas.

Lemma 4.1 *Let $\{x_n\}$ be a solution of (1.1). Then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are eventually monotone.*

Proof. We define the sequence $\{u_n\}$ and the function $h(x)$ as follows

$$u_n = x_n - A, \quad h(x) = x + A.$$

Then from (1.1) for $n \geq 3$ we get

$$\frac{u_n}{u_{n-2}} = \frac{(u_{n-2} + A)^p}{(u_{n-4} + A)^p} \frac{(u_{n-3} + A)^q}{(u_{n-1} + A)^q} = \frac{(h(u_{n-2}))^p}{(h(u_{n-4}))^p} \frac{(h(u_{n-3}))^q}{(h(u_{n-1}))^q}. \tag{4.9}$$

Then using (4.9) and arguing as in Lemma 2 of [5] (see also Theorem 2 in [20]) we can easily prove the lemma.

Lemma 4.2 *Consider equation (1.1) where (4.1) and (4.3) hold. Let x_n be a solution of (1.1) such that either*

$$A < x_{-1} < A + \epsilon_1, \quad x_0 > (A+\epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}} \tag{4.10}$$

or

$$A < x_0 < A + \epsilon_1, \quad x_{-1} > (A+\epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}. \tag{4.11}$$

Then if (4.10) hold we have

$$A < x_{2n-1} < A + \epsilon_1, \quad x_{2n} > (A+\epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}, \quad n = 0, 1, ... \tag{4.12}$$

and if (4.11) are satisfied we have

$$A < x_{2n} < A + \epsilon_1, \quad x_{2n-1} > (A+\epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}, \quad n = 0, 1, ... \tag{4.13}$$

Proof Suppose that (4.10) are satisfied. Then from (1.1) and (4.3) we have

$$A < x_1 = A + \frac{x_{-1}^p}{x_0^q} < A + \epsilon_1 \frac{(A+\epsilon_1)^p}{(A+\epsilon_1)^p} = A + \epsilon_1$$

and

$$x_2 = A + \frac{x_0^p}{x_1^q} > A + (A+\epsilon_1)^{\frac{p^2-q^2}{q}} \epsilon_1^{-\frac{p}{q}} > (A+\epsilon_1)^{\frac{p}{q}} \epsilon_1^{-\frac{1}{q}}.$$

Working inductively we can easily prove relations (4.12). Similarly if (4.11) are satisfied we can prove that (4.13) hold.

Proposition 4.2 *Consider equation (1.1) where (4.1), (4.2) and (4.3) hold. Suppose also that*

$$A + \epsilon_1 < 1. \tag{4.14}$$

Then every solution x_n of (1.1) with initial values x_{-1}, x_0 which satisfy either (4.10) or (4.11), converges to a prime two periodic solution.

Proof Let x_n be a solution with initial values x_{-1}, x_0 which satisfy either (4.10) or (4.11). Using Proposition 2.1 and Lemma 4.1 we have that there exist

$$\lim_{n\to\infty} x_{2n+1} = L, \quad \lim_{n\to\infty} x_{2n} = l.$$

In addition from Lemma 4.2 we have that either L or l belongs to the interval $(A, A+\epsilon_1)$. Furthermore from Proposition 3.1 we have that equation (1.1) has a unique equilibrium $\bar{x}$ such that $1 < \bar{x} < \infty$. Therefore from

(4.14) we have that $L \neq l$. So x_n converges to a prime two period solution. This completes the proof of the proposition.

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POSITIVE SOLUTIONS FOR NONLINEAR NEUMANN PROBLEMS WITH CONCAVE AND CONVEX TERMS

by

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a C^2-boundary $\partial\Omega$.

We consider the following nonlinear Neumann problem:

$$\begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda|u(z)|^{q-2}u(z) + f(z, u(z)) \\ \text{a.e. in } \Omega, \quad u > 0, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ \beta \in L^\infty(\Omega)_+ \setminus \{0\}, \quad \lambda > 0, \quad 1 < q < p < \infty. \end{cases} \tag{1}$$

Here $\Delta_p u = \text{div}\,(||Du||^{p-2}Du)$.

Note that the term $x \to \lambda|x|^{q-2}x$ is $(p-1)$-*sublinear* near $+\infty$, i.e.

$$\lim_{x \to +\infty} \frac{\lambda x^{q-1}}{x^{p-1}} = 0$$

("concave" term).

The Carathéodory function $f(z,x)$, $z \in \Omega$, $x \in \mathbb{R}$ is supposed to be $(p-1)$-*superlinear* near $+\infty$ in x, i.e.

$$\lim_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} = +\infty$$

("convex" perturbation).

The aim of this work is to establish a *bifurcation - type* result for the positive smooth solutions of (1), with respect to the parameter $\lambda > 0$.

Particular case: The right hand side term of (1) has the form $x \to \lambda |x|^{q-2}x + |x|^{r-2}x$, with

$$1 < q < p < r < p^* = \begin{cases} \dfrac{Np}{N-p}, & \text{if } p < N \\ +\infty, & \text{if } p \geq N \end{cases} .$$

This particular case is what we mostly encounter in the literature and only in the context of Dirichlet problems.

In this direction we mention the semilinear (i.e., $p = 2$) work of Ambrosetti-Brezis-Cerami [1], which is the first to consider problems with concave and convex terms.

The above work was extended to nonlinear problems driven by the p-Laplacian, by Garcia Azorero-Manfredi-Peral Alonso [3] and by Guo-Zhang [4], for $p \geq 2$. In the latter case, the authors also consider reactions of the form

$$\lambda |x|^{q-2}x + g(x),$$

where $g \in C^1(\mathbb{R})$, $g'(x) \geq 0$, $xg(x) \geq 0$, for $x \in \mathbb{R}$ and

$$\lim_{|x| \to 0} \frac{g(x)}{|x|^{p-2}x} = 0, \qquad \lim_{|x| \to \infty} \frac{g(x)}{|x|^{p-2}x} > \lambda_1 .$$

For Dirichlet problems driven by the p-Laplacian and with reactions of more general form we also refer to the following works:

- Boccardo-Escobedo-Peral [2]. The reaction is

  $$\lambda g(x) + x^{r-1}, \quad x \geq 0,$$

  where

  $g : \mathbb{R}_+ \to \mathbb{R}$ continuous, $g(x) \leq \hat{c}x^{q-1}$ for $x \geq 0$ with $\hat{c} > 0$, $\quad 1 < q < p < r < p^*$

  and the function $x \to \lambda g(x) + x^{r-1}$ is nondecreasing on $\mathbb{R}_+$.

  They prove the existence of only one positive solution for $\lambda > 0$ suitably small.

- Hu-Papageorgiou[5], where the "convex" ($(p-1)$-superlinear) term is a more general Caratheodory function $f(z,x)$ satisfying the well-known **Ambrosetti-Rabinowitz (AR) condition**:

  " $\exists\, \mu > p$, $M > 0$ such that $\forall x > M$,

  $$0 < \mu F(z,x) \leq f(z,x)x \quad \text{uniformly for a.a. } z \in \Omega."$$

To the best of our knowledge, no bifurcation-type results exist for the **Neumann** problem.

We mention only the work of Wu-Chen[6], where the reaction is of the form $\lambda f(z,x)$, $\lambda > 0$, $f(\cdot,\cdot)$ $(p-1)$-sublinear near infinity in $x \in \mathbb{R}$.

The authors also impose the extra restrictive conditions that $\operatorname*{essinf}_{\Omega} \beta > 0$ and that $N < p$.

They produce three solutions for all $\lambda > 0$ in an open interval. The obtained solutions are not positive.

2 The hypotheses on the perturbation.

(**H**) : The Carathéodory function $f(z,x)$, $z \in \Omega$, $x \in \mathbb{R}$ has $(r-1)$ -polynomial growth with respect to x $(p < r < p*)$. Moreover,

(i) $\lim_{x \to 0+} \dfrac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$

(ii) there exists $\delta_0 > 0$ such that

$$f(z,x) \geq 0 \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \in [0,\ \delta_0]$$

and

$\forall \theta > 0$, $\exists\, \hat{\xi}_\theta > 0$ such that for a.a. $z \in \Omega$,

$$x \to f(z,x) + \hat{\xi}_\theta x^{p-1} \quad \text{is increasing on } [0,\theta].$$

(iii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\lim_{x \to +\infty} \frac{F(z,x)}{x^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega$$

and

$$\eta_0 \leq \liminf_{x \to +\infty} \frac{f(z,x)x - pF(z,x)}{x^\tau} \quad \text{uniformly for a.a. } z \in \Omega,$$

where

$$\tau \in \left((r-p)\max\left\{1, \frac{N}{p}\right\},\ p^* \right), \quad q < \tau, \quad \eta_0 > 0$$

Remark 1: Since we are interested in positive solutions and hypotheses H (i), (ii), (iii) involve only the positive semiaxis we may assume that $f(z,x) = 0$ for a.a. $z \in Z$, all $x \leq 0$.

Remark 2: In order to express the " $(p-1)$ -superlinearity" of $f(z,x)$ with respect to x near $+\infty$, instead of the usual in such cases AR-condition, we employ the much weaker conditions H(iii).

Example:

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x^{p-1}\left(\ln(x^p+1) + \dfrac{x^p}{x^p+1} \right), & \text{if } x > 0. \end{cases}$$

Note that f satisfies H(iii) but it does not satisfy the AR-condition.

3 Some function spaces

In the study of our problem we will use the following two function spaces

$$C_n^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$$

and

$$W_n^{1,p}(\Omega) = \overline{C_n^1(\overline{\Omega})}^{||\cdot||},$$

where $||\cdot||$ denotes the Sobolev norm of $W^{1,p}(\Omega)$.
Note that $C_n^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_n^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\mathrm{int}C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

4 The Euler functional

Let $\varphi_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ be the Euler functional for problem (1) defined by

$$\varphi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \frac{\lambda}{q}||u^+||_q^q - \int_\Omega F(z,u)dz,$$

where $F(z,x) = \displaystyle\int_0^x f(z,s)ds$.

Proposition 1 *Under hypotheses (H), $\varphi_\lambda \in C^1\left(W_n^{1,p}(\Omega)\right)$ and each nontrivial critical point of φ_λ is a positive smooth solution of (1).*

The proof is mainly based on the nonlinear regularity theory and also on the nonlinear maximum principle of Vazquez combined with hypothesis H(ii):
" $\forall\theta > 0$, $\exists\, \hat{\xi}_\theta > 0$ such that for a.a. $z \in \Omega$,

$$x \to f(z,x) + \hat{\xi}_\theta x^{p-1} \quad \text{is increasing on } [0,\theta]."$$

Proposition 2 *Under hypotheses (H), φ_λ satisfies the* **Cerami** *condition (C -condition):*
" Every sequence $\{x_n\}_{n\geq 1} \subseteq X = W_n^{1,p}(\Omega)$ such that

$$\sup_n |\varphi_\lambda(x_n)| < \infty, \quad (1+||x_n||)\varphi'_\lambda(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

has a strongly convergent subsequence "

The proof crucially uses hypothesis H(iii).

5 The bifurcation -type result

$$\begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda|u(z)|^{q-2}u(z) + f(z,u(z)) \\ \text{a.e. in } \Omega, \qquad (1) \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \quad (1 < q < p < \infty). \end{cases}$$

Theorem 3 *If hypotheses (H) hold and $\beta \in L^\infty_+(\Omega) \setminus \{0\}$, then there exists $\lambda^* > 0$ such that*

(a) *for $\lambda \in (0,\ \lambda^*)$ problem (1) has at least two positive smooth solutions*

(b) *for $\lambda = \lambda^*$ problem (1) has at least one positive smooth solution*

(c) *for $\lambda > \lambda^*$ problem (1) has no positive solution*

The proof of Theorem 1 may be divided into two parts:
Part I: We consider the set

$$S = \{\lambda > 0 : \text{ problem (1) has a positive smooth } \lambda \text{ -solution}\}$$

and we prove that S is nonempty and bounded from above.
Part II: We prove that $\lambda^* = \sup S$ has the desired properties.

Sketch of the proof of Part I:

Proposition 4 *Under the hypotheses of Th. 3, there exists $\hat{\lambda} > 0$ such that for every $\lambda \in (0,\ \hat{\lambda})$ we can find $\rho_\lambda > 0$ for which we have*

$$\inf [\ \varphi_\lambda(u) : ||u|| = \rho_\lambda\ ] = \eta_\lambda > 0.$$

In order to prove Prop. 4, one shall need hypothesis H(i):

$$\text{“} \lim_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} = 0 \quad \text{uniformly for a.a. } z \in \Omega\text{”}$$

in conjunction with the $(r-1)$ -polynomial growth of $f(z,x)$ with respect to x and also with the inequalities $1 < q < p < r < p^*$.

Proposition 5 *Under the hypotheses of Th. 3, we have*

$$\varphi_\lambda(tu) \to -\infty \quad \textit{as } t \to +\infty,$$

for each $u \in C_+ \setminus \{0\}$ with $||u||_p = 1$.

The proof of Prop. 5 is based on the p-superlinearity of $F(z,x)$ with respect to x near $+\infty$ (H(iii)) and also on the fact that $q < p$.
Now Prop. 1, 2, 4, 5 via Mountain Pass Theorem yield

Proposition 6 *Under the hypotheses of Th. 3, we have $(0, \hat{\lambda}) \subseteq S$, where $\hat{\lambda}$ is as postulated in Prop. 4. Hence, $S \neq \emptyset$.*

Proposition 7 *Under the hypotheses of Th. 3, the set S is bounded from above.*

For the proof, we shall need the following

Lemma 8 *Let $\beta \in L^\infty(\Omega)_+ \setminus \{0\}$, $u, \widetilde{u} \in int\ C_+$ and $R > 0$ such that for a.a. $z \in \Omega$,*

$$-\Delta_p u(z) + \beta(z)u(z)^{p-1} + R \leq -\Delta_p \widetilde{u}(z) + \beta(z)\widetilde{u}(z)^{p-1}. \qquad (2)$$

Then $u < \widetilde{u}$ on $\overline{\Omega}$.

The proof of the above lemma is mainly based on the monotonicity properties of the operator $T : X \to X^*$ $(X = W_n^{1,p}(\Omega))$ induced by the differential operator $u \to -\Delta_p u + \beta(\cdot)|u|^{p-2}u$.

Proof of Prop. 7: The $(p-1)$ -superlinearity of $f(z,x)$ with respect to x near $+\infty$ combined with hypothesis H(ii) enables us to choose $\overline{\lambda} > 0$ large such that

$$\overline{\lambda}x^{q-1} + f(z,x) \geq ||\beta||_\infty x^{p-1} \quad \text{for a.a. } z \in \Omega, \ \text{ all } x \geq 0.$$

Claim: $\overline{\lambda}$ is an upper bound of S.

Indeed, suppose that for some $\lambda > \overline{\lambda}$ our problem has a λ -solution $u \in \text{int}\ C_+$. Let $m = \min_{\overline{\Omega}} u > 0$. Then for a.a. $z \in \Omega$,

$$\begin{aligned} -\Delta_p u(z) \ &+ \ \beta(z)u(z)^{p-1} \geq ||\beta||_\infty u(z)^{p-1} + (\lambda - \overline{\lambda})u(z)^{q-1} \\ &\geq \ -\Delta_p m + \beta(z)m^{p-1} + (\lambda - \overline{\lambda})m^{q-1} \end{aligned}$$

which implies (see Lemma 8) that $u > m$ on $\overline{\Omega}$ (false!). $\square$

<u>Sketch of the proof of Part II:</u>
<u>We begin with two Lemmas:</u>

Lemma 9 *Let $u, \widetilde{u} \in int\ C_+$ and $0 < \lambda < \widetilde{\lambda}$ such that u is a λ -solution and $\widetilde{u}$ is a $\widetilde{\lambda}$-solution. If $u \leq \widetilde{u}$, then $u < \widetilde{u}$ on $\overline{\Omega}$.*

For the proof, we set $\theta = ||\widetilde{u}||_\infty$ and we choose $\xi_\theta > 0$ such that $x \to f(z,x) + \xi_\theta x^{p-1}$ is increasing on $[0, \theta]$ (hypothesis H(ii)).

Then (2) holds for

$$\text{“}\beta(\cdot)\text{''} = \beta(\cdot) + \xi_\theta, \quad \text{“}R\text{''} = (\widetilde{\lambda} - \lambda)m^{q-1}, \quad m = \min_{\overline{\Omega}} \widetilde{u}$$

and now Lemma 8 applies.

Lemma 10 *Let $0 < \lambda < \widetilde{\lambda}$ and $\widetilde{u} \in int\ C_+$ be a $\widetilde{\lambda}$ -solution. Then there exists a λ-solution $u_0 \in int\ C_+$ such that*

$$0 < u_0 < \widetilde{u} \ \ on\ \overline{\Omega}, \quad \varphi_\lambda(u_0) < 0.$$

Proof: We consider the following truncation of the reaction:

$$g_\lambda(z,x) = \begin{cases} 0, & \text{if} \quad x \le 0 \\ \lambda x^{q-1} + f(z,x), & \text{if} \quad 0 < x < \widetilde{u}(z) \\ \lambda\, \widetilde{u}(z)^{q-1} + f(z, \widetilde{u}(z)), & \text{if} \quad \widetilde{u}(z) \le x. \end{cases}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the C^1-functional $\psi_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \int_\Omega G_\lambda(z,u)dz.$$

By using suitable test functions we may show that each critical point of ψ_λ lies in the interval $[0, \widetilde{u}]$ and it is also a critical point of the Euler functional φ_λ.

Note that ψ_λ is **coercive** and weakly lower semicontinuous, so we can find $u_0 \in W_n^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_0) = \inf[\, \psi_\lambda(u) : u \in W_n^{1,p}(\Omega)\,].$$

Then $\psi_\lambda'(u_0) = 0 \Rightarrow u_0 \in [0, \widetilde{u}]$ and $\varphi_\lambda'(u_0) = 0$.
Moreover, we may show that for sufficiently small
$t > 0$, we have $\psi_\lambda(t) < 0$, so

$$\psi_\lambda(u_0) < 0 = \psi_\lambda(0) \Rightarrow u_0 \neq 0.$$

It follows that u_0 is a positive smooth λ-solution with $\varphi_\lambda(u_0) = \psi_\lambda(u_0) < 0$.
Finally, since $\lambda < \widetilde{\lambda}$, we have $u_0 < \widetilde{u}$ (see Lemma 5).
Thus, $u_0 \in (0, \widetilde{u})$. □

To proceed, set $\lambda^* = \sup S$.

Proposition 11 *If hypotheses of Th. 3 hold and* $\lambda \in (0, \lambda^*)$, *then problem (1) has least two smooth positive solutions*

$$u_0\,,\ \hat{u} \in intC_+\,, \quad u_0 \neq \hat{u}, \quad u_0 \le \hat{u}, \quad \varphi_\lambda(u_0) < 0.$$

Sketch of the proof:
Let $\lambda \in (0, \lambda^*)$. Choose $\widetilde{\lambda} \in (\lambda,\ \lambda^*) \cap S$ and a $\widetilde{\lambda}$ -solution $\widetilde{u} \in$ int C_+ .
By view of Lemma 10, we may find a λ -solution $u_0 \in$ int C_+ such that

$$0 < u_0 < \widetilde{u}, \quad \varphi_\lambda(u_0) < 0.$$

Next, consider the following truncation of the reaction:

$$\hat{f}_\lambda(z,x) = \begin{cases} \lambda u_0(z)^{q-1} + f(z, u_0(z)), & \text{if } x \le u_0(z) \\ \lambda x^{q-1} + f(z,x), & \text{if } u_0(z) < x. \end{cases}$$

Let $\hat{F}_\lambda(z,x) = \displaystyle\int_0^x \hat{f}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \int_\Omega \hat{F}_\lambda(z,u)dz.$$

By using suitable test functions we may show that for each critical point w of $\hat{\varphi}_\lambda$, we have $u_0 \leq w$ and that w is also a critical point of the Euler functional φ_λ.

Evidently, $\hat{\varphi}_\lambda|_{[0,\, \widetilde{u}]}$ is coercive and weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in [0,\, \widetilde{u}]$ such that

$$\hat{\varphi}_\lambda(\widetilde{u}_0) = \inf[\, \hat{\varphi}_\lambda(u) : u \in [0,\, \widetilde{u}]\,]\,.$$

Then

$$-\hat{\varphi}'_\lambda(\widetilde{u}_0) \in N_{[\,0,\, \widetilde{u}\,]}(\widetilde{u}_0)$$

where $N_{[\,0,\, \widetilde{u}\,]}(\widetilde{u}_0)$ denotes the normal cone to $[\,0,\, \widetilde{u}\,]$ at $\widetilde{u}_0$.

By using the definition of the normal cone of a closed and convex set combined with our hypotheses, we may show that $\hat{\varphi}'_\lambda(\widetilde{u}_0) = 0$.

It follows that $u_0 \leq \widetilde{u}_0$ and that $\widetilde{u}_0$ is a nontrivial critical point of the Euler functional φ_λ. Hence, $\widetilde{u}_0$ is also a positive smooth λ-solution to our problem.

- If $\widetilde{u}_0 \neq u_0$, we are done.
- Suppose that $\widetilde{u}_0 = u_0$. Since $u_0 \in (0, \widetilde{u})$, we infer that

$$u_0 \quad \text{is a local } \; C^1_n(\overline{\Omega}) \; - \text{minimizer of } \hat{\varphi}_\lambda\,.$$

It follows from a fact due to Barletta -Papageorgiou (which extends previous results of Brezis - Nirenberg and of Azorero-Manfredi-Alonso) that

$$u_0 \quad \text{is a local } \; W^{1,p}_n(\Omega) \; - \text{minimizer of } \hat{\varphi}_\lambda\,.$$

Without loss of generality, we may assume that u_0 is an isolated critical point and local minimizer of the functional $\hat{\varphi}_\lambda$.

Then we prove that:

- for some $\rho > 0$,

$$\hat{\varphi}_\lambda(u_0) < \inf[\, \hat{\varphi}_\lambda(u) : ||u - u_0|| = \rho\,]$$

- for every $u \in$ int C_+ with $||u||_p = 1$,

$$\hat{\varphi}_\lambda(tu) \to -\infty, \quad \text{as } t \to +\infty$$

- $\hat{\varphi}_\lambda$ satisfies the C -condition

Arguing via Mountain Pass Theorem we may find a critical point $\hat{u}$ of $\hat{\varphi}_\lambda$ such that $\hat{u} \neq u_0$.
It follows that $u_0 \leq \hat{u}$ and that $\hat{u}$ is a nontrivial critical point of the Euler functional φ_λ.
Hence, $\hat{u}$ is a second positive smooth λ-solution to our problem. □

Proposition 12 *If hypotheses of Th. 3 hold, then for $\lambda = \lambda^*$, problem (1) has at least one smooth positive solution.*

The key ingredient in the proof of Proposition 12, is the following

Lemma 13 *Let $S' \subseteq S$ be nonempty and bounded from below with* $\inf S' > 0$ *and $B \subseteq$ int C_+ be $\|\cdot\|_\infty$ -bounded. Then there exists $w \in$ int C_+ such that for each $\lambda \in S'$ and for each λ-solution $u \in B$, we have $w \leq u$.*

Sketch of the proof of Prop. 12: Choose a nondecreasing sequence $(\lambda_n) \subseteq S$ such that $\lambda_n \uparrow \lambda^*$. By view of Prop.11, we may find $\{u_n\}_{n\geq 1} \subseteq$ int C_+ such that

$$\varphi'_{\lambda_n}(u_n) = 0, \quad \varphi_{\lambda_n}(u_n) < 0, \quad \text{for all } n \geq 1.$$

Arguing in a similar way as in the proof of the Cerami condition, we may show (by passing to subsequences) that

$$u_n \to u_*, \text{ strongly in } W_n^{1,p}(\Omega).$$

Then nonlinear regularity theory guarantees that

$$\sup_n \|u_n\|_\infty < \infty$$

and that u_* is a smooth λ^*-solution.

Now Lemma 13 asserts that for some $w \in$ int C_+, we have $w \leq u_n$, $n \geq 1$. Thus, $w \leq u_*$, so $u_* \in$ int C_+.

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OSCILLATION CRITERIA FOR FIRST AND SECOND-ORDER DIFFERENCE EQUATIONS

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ABSTRACT

Consider the first-order and the second-order delay difference equations

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1}$$

and

$$\Delta^2 x(n) + p(n)x(\tau(n)) = 0, \; n = 0, 1, 2, ..., \tag{2}$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \to \mathbb{R}^+$, $\tau : \mathbb{N} \to \mathbb{N}$, $\tau(n) \leq n-1$ and $\lim_{n\to\infty} \tau(n) = +\infty$,

The most interesting oscillation criteria for Eq.(1), and Eq. (2),especially in the case where

$$0 < \liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) \leq \frac{1}{e} \text{ and } \limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) < 1.$$

for Eq.(1), are presented.

1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions of the first-order delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1}$$

has been the subject of many investigations, especially in the case where the delay $n - \tau(n)$ is a constant, that is, in the special case of the difference equation

$$\Delta x(n) + p(n)x(n-k) = 0, \;\; n = 0, 1, 2, ... \tag{1$'$}$$

[0] *Key Words:* Oscillation; delay, difference, differential equations.
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The oscillation theory of the second-order delay difference equation

$$\Delta^2 x(n) + p(n)x(\tau(n)) = 0, \tag{2}$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \to \mathbb{R}^+$, $\tau : \mathbb{N} \to \mathbb{N}$, k is a positive integer, $\tau(n) \leq n-1$ and $\lim_{n\to\infty} \tau(n) = +\infty$, has also attracted growing attention in the recent few years. See, for example, [1, 2, 4–16, 18, 19, 24, 26, 30, 36, 40, 42, 43, 45-47, 52-55, 58-64, 68, 69, 71–76] and the references cited therein.

Strong interest in Eq.(1), Eq.(1)$'$, and Eq. (2), are motivated by the fact that they represent discrete analogues of the delay differential equations

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1_c}$$

$$x'(t) + p(t)x(t-\tau) = 0, \ \ \tau > 0. \tag{1_c)$'$ ($$$$

and

$$x''(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{2_c}$$

respectively, where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$. See [3, 17, 20-23, 25, 27-29, 31-35, 37-39, 41, 44, 48-51, 56, 57, 65-67, 70] and the references cited therein.

By a solution of Eq.(1) we mean a sequence $x(n)$ which is defined for $n \geq \min\{\tau(n) : n \geq 0\}$ and which satisfies Eq.(1) for all $n \geq 0$. A solution $x(n)$ of Eq.(1) is said to be *oscillatory* if the terms $x(n)$ of the solution are neither eventually positive nor eventually negative. Otherwise the solution is called *nonoscillatory*. (Analogously for Eq.(1)$'$and Eq.(2))

In this paper our purpose is to present the state of the art on the oscillation of all solutions to Eq.(1), Eq. (1)$'$ and Eq. (2), especially in the case where

$$0 < \liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) \leq \frac{1}{e} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) < 1.$$

for Eq.(1), and

$$0 < \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) < 1$$

for Eq.(1)$'$.

2 Oscillation Criteria for Eq. (1)′

In this section we study the difference equation

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n = 0, 1, 2, \ldots \tag{1'}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and k is a positive integer.

In 1981, Domshlak [14] was the first who studied this problem in the case where $k = 1$. Then, in 1989, Erbe and Zhang [24] established that all solutions of Eq.(1)′ are oscillatory if

$$\liminf_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \tag{2.1}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1. \tag{C_1'}$$

In the same year, 1989, Ladas, Philos and Sficas [43] proved that a sufficient condition for all solutions of Eq.(1)′ to be oscillatory is that

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1} \tag{C_2'}$$

Therefore they improved the condition (2.1) by replacing the $p(n)$ of (2.1) by the arithmetic mean of $p(n-k), \ldots, p(n-1)$ in $(C_2)'$.

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (2.1) it should be emphasized that, as it is shown in [24], if

$$\sup p(n) < \frac{k^k}{(k+1)^{k+1}}$$

then Eq.(1)′ has a nonoscillatory solution.

In 1990, Ladas [42] conjectured that Eq.(1)′ has a nonoscillatory solution if

$$\sum_{i=n-k}^{n-1} p(i) < \left(\frac{k}{k+1}\right)^{k+1}$$

holds eventually. However, a counterexample to this conjecture was given in 1994, by Yu, Zhang and Wang [73].

It is interesting to establish sufficient oscillation conditions for the equation (1)′ in the case where neither $(C_1)'$ nor $(C_2)'$ is satisfied.

In 1995, the following oscillation criterion was established by Stavroulakis [54]:

Theorem 2.1 ([54]) *Assume that*

$$\alpha_0 := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) \le \left(\frac{k}{k+1}\right)^{k+1}$$

and

$$\limsup_{n\to\infty} p(n) > 1 - \frac{\alpha_0^2}{4} \tag{2.2}$$

then all solutions of Eq.$(1)'$ *oscillate.*

In 2004, the same author [55] improved the condition (2.2) as follows:

Theorem 2.2 ([55]) *If* $0 < \alpha_0 \le \left(\frac{k}{k+1}\right)^{k+1}$, *then either one of the conditions*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{4} \tag*{$(C_3)'$}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \alpha_0^k \tag{2.3}$$

implies that all solutions of Eq.$(1)'$ *oscillate.*

In 2006, Chatzarakis and Stavroulakis [8], established the following

Theorem 2.3 ([8]) *If* $0 < \alpha_0 \le \left(\frac{k}{k+1}\right)^{k+1}$ *and*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{2(2-\alpha_0)} \tag{2.4}$$

then all solutions of Eq.$(1)'$ *oscillate.*

Remark 2.1. Observe the following:

(i) When $\alpha \longrightarrow 0$, then it is clear that the conditions $(C_3)'$, (2.3) and (2.4) reduce to

$$\mathrm{A} := \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1,$$

which obviously implies $(C_1)'$.

(ii) It always holds

$$\frac{\alpha^2}{2(2-\alpha)} > \frac{\alpha^2}{4},$$

sinca $\alpha > 0$ and therefore condition $(C_3)'$ always implies (2.4).

(iii) When $k = 1, 2$

$$\frac{\alpha^2}{2(2-\alpha)} < \alpha^k,$$

(since, from the above mentioned conditions, it makes sense to investigate the case when $\alpha \leq \left(\left(\frac{k}{k+1}\right)^{k+1}\right)$ and therefore condition (2.4) implies (2.3).

(iv) When $k = 3$,

$$\frac{\alpha^2}{2(2-\alpha)} > \alpha^3 \text{ if } 0 < \alpha < 1 - \frac{\sqrt{2}}{2}$$

while

$$\frac{\alpha^2}{2(2-\alpha)} < \alpha^3 \text{ if } 1 - \frac{\sqrt{2}}{2} < \alpha \leq \left(\frac{3}{4}\right)^4.$$

So in this case the conditions (2.4) and (2.3) are independent.

(v) When $k \geq 4$

$$\frac{\alpha^2}{2(2-\alpha)} > \alpha^k,$$

and therefore condition (2.3) implies (2.4).

(vi) When $k \geq 10$ condition (2.4) may hold but condition $(C_1)'$ may not hold.

(vii) When k is large then $\alpha \longrightarrow \frac{1}{e}$ and in this case both conditions $(C_3)'$ and (2.3) imply (2.4). For illustartive purposes, we give the values of the lower bound of A under these conditions when $k = 100$ ($\alpha \simeq 0.366$) :

(2.3)	:	0.999999
$(C_3)'$	:	0.966511
(2.4)	:	0.959009

We see that the condition (2.4) essentially improves the conditions $(C_3)'$ and (2.3).

Also, Chen and Yu [9] obtained the following oscillation condition

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1 - \frac{1-\alpha_0-\sqrt{1-2\alpha_0-\alpha_0^2}}{2}. \qquad (C_4)'$$

3 Oscillation Criteria for Eq. (1)

In this section we study the difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n-1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$.

In the case of Eq.(1) with a general delay argument $\tau(n)$, from Chatzarakis, Koplatadze and Stavroulakis [4], it follows the following

Theorem 3.1 ([4]) *If*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) > 1 \tag{C_1}$$

then all solutions of Eq. (1) *oscillate.*

This result generalizes the oscillation criterion $(C_1)'$. Also Chatzarakis, Koplatadze and Stavroulakis [5] extended the oscillation criterion $(C_2)'$ to the general case of Eq. (1). More precisely, the following theorem has been established in [5].

Theorem 3.2 ([5]) *Assume that*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) < +\infty \tag{3.1}$$

and

$$\alpha := \liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) > \frac{1}{e}. \tag{C_2}$$

Then all solutions of Eq.(1) *oscillate.*

Remark 3.1 It is to be pointed out that the conditions (C_1) and (C_2) are the discrete analogues of the conditions $(C_1)'$ and $(C_2)'$ for Eq.(1) in the case of a general delay argument $\tau(n)$.

Remark 3.2 ([5]). The condition (C_2) is optimal for Eq.(1) under the assumption that $\lim\limits_{n\to+\infty}(n-\tau(n)) = \infty$, since in this case the set of natural numbers increases infinitely in the interval $[\tau(n), n-1]$ for $n \to \infty$.

Now, we are going to present an example to show that the condition (C_2) is optimal, in the sense that it cannot be replaced by the non-strong inequality.

Example 3.1 ([5]) Consider Eq.(1), where

$$\tau(n) = [\beta n],\ p(n) = \left(n^{-\lambda} - (n+1)^{-\lambda}\right)([\beta n])^{\lambda},\ \beta \in (0,1),\ \lambda = -\ln^{-1}\beta \tag{3.2}$$

and $[\beta n]$ denotes the integer part of βn.

It is obvious that

$$n^{1+\lambda}\left(n^{-\lambda}-(n+1)^{-\lambda}\right)\rightarrow\lambda \quad for \quad n\rightarrow\infty.$$

Therefore

$$n\left(n^{-\lambda}-(n+1)^{-\lambda}\right)([\beta n])^{\lambda}\rightarrow\frac{\lambda}{e} \quad for \quad n\rightarrow\infty. \tag{3.3}$$

Hence, in view of (3.2) and (3.3), we have

$$\begin{aligned}\liminf_{n\rightarrow\infty}\sum_{i=\tau(n)}^{n-1}p(i) &= \frac{\lambda}{e}\liminf_{n\rightarrow\infty}\sum_{i=[\beta n]}^{n-1}\frac{e}{\lambda}i\left(i^{-\lambda}-(i+1)^{-\lambda}\right)([\beta i])^{\lambda}\cdot\frac{1}{i}\\ &= \frac{\lambda}{e}\liminf_{n\rightarrow\infty}\sum_{i=[\beta n]}^{n-1}\frac{1}{i}=\frac{\lambda}{e}\ln\frac{1}{\beta}=\frac{1}{e}\end{aligned}$$

or

$$\liminf_{n\rightarrow\infty}\sum_{i=\tau(n)}^{n-1}p(i)=\frac{1}{e}. \tag{3.4}$$

Observe that all the conditions of Theorem 3.2 are satisfied except the condition (C_2). In this case it is not guaranteed that all solutions of Eq.(1) oscillate. Indeed, it is easy to see that the function $u=n^{-\lambda}$ is a positive solution of Eq.(1).

As it has been mentioned above, it is an interesting problem to find new sufficient conditions for the oscillation of all solutions of the delay difference equation (1), in the case where neither (C_1) nor (C_2) is satisfied.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [4] investigated for the first time this question for Eq.(1) in the case of a general delay argument $\tau(n)$ and derived the following theorem.

Theorem 3.3 ([4]) *Assume that* $0<\alpha\leq\frac{1}{e}$. *Then we have:*

(I) *If*

$$\limsup_{n\rightarrow\infty}\sum_{j=\tau(n)}^{n}p(j)>1-\left(1-\sqrt{1-\alpha}\right)^2 \tag{3.5}$$

then all solutions of Eq.(1) *oscillate.*

(II) *If in addition,*

$$p(n)\geq 1-\sqrt{1-\alpha} \text{ for all large } n, \tag{3.6}$$

and

$$\limsup_{n\rightarrow\infty}\sum_{j=\tau(n)}^{n}p(j)>1-\alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}} \tag{3.7}$$

then all solutions of Eq.(1) *oscillate.*

Recently, the above result was improved in [6] and [7] as follows:

Theorem 3.4 ([6]) (I) *If* $0 < \alpha \leq \frac{1}{e}$ *and*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right) \tag{3.8}$$

then all solutions of Eq.(1) *oscillate.*

(II) *If* $0 < \alpha \leq 6 - 4\sqrt{2}$ *and in addition,*

$$p(n) \geq \frac{\alpha}{2} \text{ for all large } n, \tag{3.9}$$

and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) \tag{3.10}$$

then all solutions of Eq.(1) *are oscillatory.*

Theorem 3.5 ([7]) *Assume that* $0 < \alpha \leq -1 + \sqrt{2}$, *and*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right) \tag{C_4}$$

then all solutions of Eq.(1) *oscillate.*

Remark 3.3 Observe the following:

(i) When $0 < \alpha \leq \frac{1}{e}$, it is easy to verify that

$$\frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} > \alpha\frac{1 - \sqrt{1-\alpha}}{\sqrt{1-\alpha}} > \frac{1 - \alpha - \sqrt{1-2\alpha}}{2} > (1 - \sqrt{1-\alpha})^2$$

and therefore the condition (C_4) is weaker than the conditions (3.7), (3.8) and (3.5).

(ii) When $0 < \alpha \leq 6 - 4\sqrt{2}$, it is easy to show that

$$\frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) > \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right),$$

and therefore in this case and when (3.9) holds, inequality (3.10) improves the inequality (C_4) and especially, when $\alpha = 6 - 4\sqrt{2} \simeq 0.3431457$, the lower bound in (C_4) is 0.8929094 while in (3.10) is 0.7573593.

4 Oscillation Criteria for Eq. (2)

In this section we study the second-order difference equation

$$\Delta^2 x(n) + p(n)x(\tau(n)) = 0 \tag{2}$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \to \mathbb{R}_+$, $\tau : \mathbb{N} \to \mathbb{N}$, $\tau(n) \le n-1$ and $\lim_{n\to\infty} \tau(n) = +\infty$.

In 1994, Wyrwinska [69] proved that all solutions of Eq. (2) are oscillatory if

$$\limsup_{n\to\infty} \left\{ \sum_{i=\tau(n)}^{n} [\tau(i) - 2]p(i) + [\tau(n) - 2] \sum_{i=n+1}^{\infty} p(i) \right\} > 1,$$

while, in 1997, Agarwal, Thandapani and Wong [1] proved that, in the special case of the second-order difference equation with constant delay

$$\Delta^2 x(n) + p(n)x(n-k) = 0, \quad k \ge 1 \tag{2_c}'$$

all solutions are oscillatory if

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} (i-k)p(i) > 2\left(\frac{k}{k+1}\right)^{k+1}.$$

In 2001, Grzecorczyk and Werbowski [26] studied Eq.$(2_c)'$ and proved that under the following conditions

$$\limsup_{n\to\infty} \left\{ \begin{array}{c} \sum_{i=n-k}^{n}(i-n+k+1)p(i)+ \\ +\left[(n-k-2)+\sum_{i=n_1}^{n-k-1}(i-k)^2 p(i)\right] \times \\ \times \sum_{i=n+1}^{\infty} p(i) \end{array} \right\} > 1, \text{ for some } n_1 > n_0,$$

or

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} (i-k-1)p(i) > \left(\frac{k}{k+1}\right)^{k+1} \tag{C_2}''$$

all solutions of Eq. $(2_c)'$are oscillatory. Observe that the last condition $(C_2)''$ may be seen as the discrete analogue of the condition

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > \frac{1}{e}$$

for Eq. (2_c).

In 2001 Koplatadze [36] studied the oscillatory behaviour of all solutions to the equation (2) with variable delay and established the following.

Theorem 4.1 ([36]) *Assume that*

$$\inf\left\{\frac{1}{1-\lambda}\liminf_{n\to\infty} n^{-\lambda}\sum_{i=1}^{n} ip(i)\tau^{\lambda}(i) : \lambda\in(0,1)\right\} > 1$$

and

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} ip(i)\tau(i) > 0.$$

Then all solutions of Eq.(2) *oscillate.*

Corollary 4.1 ([36]) *Let* $\alpha > 0$ *and*

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} i^2 p(i) > \max\left\{\alpha^{-\lambda}\lambda(1-\lambda) : \lambda\in[0,1]\right\}.$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x([\alpha n]) = 0, \quad n \geq \max\left\{1, \frac{1}{\alpha}\right\}, \quad n\in N$$

oscillate.

Corollary 4.2 ([36]) *Let* n_0 *be an integer and*

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} i^2 p(i) > \frac{1}{4}.$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x(n-n_0) = 0, \quad n \geq \max\{1, n_0+1\}, \quad n\in N$$

oscillate.

In 2002 Koplatadze, Kvinikadze and Stavroulakis [40] studied Eq.(2) and established the following.

Theorem 4.2 ([40]) *Assume that*

$$\liminf_{n\to\infty}\frac{\tau(n)}{n} = \alpha\in(0,\infty),$$

and

$$\liminf_{n\to\infty} n\sum_{i=n}^{\infty} p(i) > \max\left\{\alpha^{-\lambda}\lambda(1-\lambda) : \lambda\in[0,1]\right\}. \tag{4.1}$$

Then all solutions of Eq.(2) *oscillate.*

In the case where $\alpha = 1$, the following discrete analogue of Hille's well-known oscillation theorem for 2nd order ordinary differential equations (see [29]) is derived.

Theorem 4.3 ([40]) *Let* n_0 *be an integer and*

$$\liminf_{n\to\infty} n \sum_{i=n}^{\infty} p(i) > \frac{1}{4}. \tag{4.2}$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x(n - n_0) = 0, \quad n \geq n_0,$$

oscillate.

Remark 4.1 ([40]) As in case of ordinary differential equations, the constant $1/4$ in (4.2) is optimal in the sense that the strict inequality cannot be replaced by the nonstrict one. More than that, the same is true for the condition (4.1) as well. To ascertain this, denote by c the right-hand side of (4.1), and by λ_0 the point where the maximum is achieved. The sequence $x(n) = n^{\lambda_0}$ obviously is a nonoscillatory solution of the equation

$$\Delta^2 x(n) + p(n)x([\alpha n]) = 0,$$

where $p(n) = -\Delta^2(n^{\lambda_0})/[\alpha n]^{\lambda_0}$ and $[\alpha]$ denotes the integer part of α. It can be easily calculated that

$$p(n) = -\frac{c}{n^2} + o\left(\frac{1}{n^2}\right) \quad \text{as} \quad n \to \infty.$$

Hence for arbitrary $\varepsilon > 0$, $p(n) \geq (c-\varepsilon)/\,n^2$ for $n \in \mathbb{N}_{\kappa_\mu}$ with $n_0 \in \mathbb{N}$ sufficiently large. Using the inequality $\sum_{i=n}^{\infty} i^2 \geq n^{-1}$ and the arbitrariness of ε, we obtain

$$\liminf_{n\to\infty} n \sum_{i=n}^{\infty} p(i) \geq c.$$

This limit can not be greater than c by Theorem 4.2. Therefore it equals c and (4.1) is violated.

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Η δομή των δεύτερων δυϊκών Καθολικά Αδιάσπαστων Banach αλγεβρών και οι άλγεβρες διαγωνίων τελεστών

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Μη τεριμμένη διάσπαση ενός χώρου Banach X, ονομάζεται η γραφή $X = Y \oplus Z$ με τους Y, Z να είναι απειροδιάστατοι. Ένας απειροδιάστατος χώρος Banach X λέγεται αδιάσπαστος αν δεν επιδέχεται μη τεριμμμένη διάσπαση. Ο X λέγεται Καθολικά Αδιάσπαστος [ή κληρονομικά αδιάσπαστος, Hereditarily Indecomposable, (H.I.)] αν κανένας απειροδιάστατος υπόχωρός του δεν επιδέχεται μη τεριμμένη διάσπαση. Ένας χαρακτηρισμός των H.I. χώρων είναι ο εξής: Ο X είναι H.I. αν και μόνο αν για κάθε ζεύγος Y, Z απειροδιάστατων υποχώρων του X και για κάθε $\varepsilon > 0$ υπάρχουν $y \in Y$, $z \in Z$ με $\|y\| = \|z\| = 1$ ώστε $\|y - z\| < \varepsilon$.

Η έννοια ορίστηκε το 1992 στο άρθρο των W.T. Gowers και B. Maurey [6] όπου κατασκευάστηκε το πρώτο παράδειγμα χώρου με αυτή την ιδιότητα. Επίσης στο ίδιο άρθρο δείχθηκε ότι για κάθε μιγαδικό H.I. χώρο X, κάθε φραγμένος γραμμικός τελεστής $T : X \to X$ είναι της μορφής $T = \lambda I + S$ με $\lambda \in \mathbb{C}$ και S ένα strictly singular τελεστή. Αυτό έχει ως συνέπεια ότι για κάθε H.I. χώρο (πραγματικό ή μιγαδικό), ο χώρος δεν είναι ισόμορφος με κανένα γνήσιο υπόχωρό του.

Από τότε η κλάση των H.I. χώρων και η θέση της μέσα στη θεωρία των χώρων Banach έχει μελετηθεί εκτενώς από πολλούς ερευνητές. Αναφέρουμε ενδεικτικά κάποια αποτελέσματα. Ο W.T. Gowers το 1996 στην περίφημη διχοτομία του, ([7]) απέδειξε ότι κάθε χώρος Banach περιέχει είτε έναν υπόχωρο με unconditional βάση ή έναν H.I. υπόχωρο. Ο Σ. Αργυρός το 2001 ([1]) έδειξε ότι αν ένας διαχωρίσιμος χώρος Banach περιέχει ισομορφικό αντίγραφο κάθε αυτοπαθούς χώρου τότε περιέχει (τον $C[0,1]$ και άρα) ισομορφικό αντίγραφο κάθε διαχωρισίμου χώρου Banach. Οι Σ. Αργυρός και Β. Φελουζής απέδειξαν ([3], 2000) ότι κάθε χώρος Banach περιέχει είτε τον $\ell_1(\mathbb{N})$ ή έναν υπόχωρο που είναι πηλίκο ενός H.I. χώρου. Οι Σ. Αργυρός και Θ. Ραϊκόφτσαλης ([4], 2010) έδειξαν ότι κάθε αυτοπαθής διαχωρίσιμος χώρος Banach είναι πηλίκο ενός αυτοπαθούς H.I. χώρου. Άρα κάθε αυτοπαθής διαχωρίσιμος χώρος περιέχεται σε έναν αυτοπαθή αδιάσπαστο χώρο. Το 2004 ([5]) κατασκευάστηκε από τους Σ. Αργυρό και Α. Τόλια ο πρώτος μη διαχωρίσιμος H.I χώρος. Επίσης, μεταξύ άλλων, αποδείχθηκε η πλήρης διχοτομία για πηλίκα H.I. χώρων. Συγκεκριμένα αν Z είναι ένας διαχωρίσιμος χώρος Banach που δεν περιέχει τον $\ell_1(\mathbb{N})$, τότε ο Z είναι πηλίκο ενός H.I. χώρου, δηλαδή υπάρχει H.I. χώρος X και τελεστής πηλίκο (δηλ. επί) $Q : X \to Z$ και μάλιστα ο Z^* περιέχεται στον X^* ως συμπληρωματικός υπόχωρος. Από την κατασκευή του, ο X έχει μια boundedly complete Schauder βάση $(e_n)_{n\in\mathbb{N}}$. Στην περίπτωση που ο Z έχει Schauder βάση, ο X^* είναι ισομορφικός με το ευθύ άθροισμα $Z^* \oplus X_*$ όπου $X_* = \overline{\text{span}}\{e_n^* : n \in \mathbb{N}\}$ είναι ο προδυϊκός του X και είναι επίσης H.I. Στην παρούσα εργασία θα εξετάσουμε τι μπορούμε επιπλέον να επιτύχουμε για τους X_*, X, X^* στην περίπτωση που ο δυϊκός χώρος Z^* έχει τη δομή Banach άλγεβρας.

Όταν X είναι ένας χώρος Banach με Schauder βάση $(e_n)_{n\in\mathbb{N}}$, ένας φραγμένος γραμμικός τελεστής $T : X \to X$ θα λέγεται διαγώνιος, αν υπάρχει $(\lambda_n)_{n\in\mathbb{N}}$ ακολουθία βαθμωτών, ώστε $T(e_n) = \lambda_n e_n$ για κάθε $n \in \mathbb{N}$. Με $\mathcal{L}_{\text{diag}}(X)$ συμβολίζουμε την άλγεβρα των φραγμένων γραμμικών τελεστών $T : X \to X$ που είναι διαγώνιοι. Συμβολίζουμε με $\overline{e}_n$ τον διαγώνιο τελεστή $\overline{e}_n = e_n^* \otimes e_n$, δηλαδή το διαγώνιο τελεστή $\overline{e}_n : X \to X$ που ορίζεται από τον τύπο $\overline{e}_n(\sum_{i=1}^{\infty} \lambda_i e_i) = \lambda_n e_n$. Οι Σ. Αργυρός, Ε. Δεληγιάννη και Α. Τόλιας απέδειξαν το εξής: Αν X είναι χώρος Banach με μια νορμαρισμένη μονότονη Schauder βάση $(e_n)_{n\in\mathbb{N}}$ και $C_1, C_2 > 0$, τα ακόλουθα είναι ισοδύναμα:

(1) Ο τελεστής

$$\begin{array}{rccc} \Phi : & X^* & \longrightarrow & \mathcal{L}_{\mathrm{diag}}(X) \\ & \sum_{n=1}^{\infty} \lambda_n e_n^* & \longmapsto & \sum_{n=1}^{\infty} \lambda_n \overline{e}_n \end{array}$$

είναι καλά ορισμένος επί και ισομορφισμός,
με $C_1 \cdot \|f\| \leq \|\Phi(f)\| \leq C_2 \cdot \|f\|$ για κάθε $f \in X^*$.
(Η σύγκλιση της σειράς στον X^* λαμβάνεται ως προς την w^* τοπολογία, ενώ στον $\mathcal{L}_{\mathrm{diag}}(X)$ ως προς την strong operator topology).

(2) **(2α)** $C_1 \cdot |\sum_{i=1}^{n} \mu_i| \leq \|\sum_{i=1}^{n} \mu_i e_i\|$ για κάθε $n \in \mathbb{N}$ και $\mu_1, \cdots, \mu_n \in \mathbb{R}$.

(2β) $\|\sum_{i=1}^{n} a_i \beta_i e_i^*\| \leq C_2 \cdot \|\sum_{i=1}^{n} a_i e_i^*\| \cdot \|\sum_{i=1}^{n} \beta_i e_i^*\|$ για κάθε $n \in \mathbb{N}$ και $a_1, \beta_1, \cdots, a_n, \beta_n \in \mathbb{R}$.

(3) Υπάρχει ένα norming σύνολο K του X ώστε:

(3α) $\pm C_1 \sum_{i=1}^{n} e_i^* \in K$ για κάθε n.

(3β) $K \cdot K \subset C_2 \cdot B_{X^*}$.

(δηλ. αν $\sum_{i=1}^{n} a_i e_i^*, \sum_{i=1}^{m} \beta_i e_i^* \in K$ τότε $\|\sum_{i=1}^{\min\{n,m\}} a_i \beta_i e_i^*\| \leq C_2$).

Με χρήση του θεωρήματος αυτού επιτεύχθηκε η κατασκευή μιας άλγεβρας διαγωνίων τελεστών επί ενός χώρου Banach X με βάση $(e_n)_{n\in\mathbb{N}}$, ώστε η άλγεβρα διαγωνίων τελεστών $\mathcal{L}_{\mathrm{diag}}(X)$ να είναι (ισομετρική της άλγεβρας X^* και) H.I.

Στην παρούσα εργασία εξετάζουμε τη δομή των δεύτερων δυϊκών H.I Banach αλγεβρών. Πιο συγκεκριμένα έχουμε το εξής θεώρημα.

Θεώρημα 1. Έστω Z ένας χώρος Banach που ικανοποιεί τα εξής:

(i) Ο Z έχει μια νορμαρισμένη διμονότονη βάση Schauder $(z_i)_{i\in\mathbb{N}}$. Θα συμβολίζουμε με $(z_i^*)_{i\in\mathbb{N}}$ τα διορθογώνια συναρτησοειδή της βάσης αυτής.

(ii) Ο Z^* είναι Banach άλγεβρα ως προς τα κατά σημείο γινόμενα. Δηλαδή αν $f, g \in Z^*$, $f = w^* - \sum_{i=1}^{\infty} \lambda_i z_i^*$, $g = w^* - \sum_{i=1}^{\infty} \mu_i z_i^*$ τότε η σειρά $f \cdot g = \sum_{i=1}^{\infty} \lambda_i \mu_i z_i^*$ συγκλίνει στη w^* τοπολογία στον Z^* και $\|f \cdot g\|_{Z^*} \leq \|f\|_{Z^*} \cdot \|g\|_{Z^*}$.

(iii) Ο Z δεν περιέχει ισομορφικά τον $\ell_1(\mathbb{N})$.

Τότε υπάρχει ένας χώρος Banach X με boundedly complete Schauder βάση $(e_n)_{n\in\mathbb{N}}$ ώστε

(α) Ο X είναι H.I. χώρος.

(β) Ο $X_* = \overline{\mathrm{span}}\{e_n^* : n \in \mathbb{N}\}$, που είναι προδυϊκός του X, είναι H.I. Banach άλγεβρα (με το κατά σημείο γινόμενο ως προς την $(e_n^*)_{n\in\mathbb{N}}$).

(γ) Υπάρχει τελεστής $Q : X \to Z$ επί και μάλιστα ο X^* είναι ισομορφικός με το ευθύ άθροισμα $Z^* \oplus X_*$.

Αν επιπλέον υποθέσουμε ότι η βάση $(z_i)_{i\in\mathbb{N}}$ είναι υποσυμμετρική, τότε ο X μπορεί να επιλεγεί ώστε να ισχύει επιπλέον το εξής:

(δ) Ο X^* είναι ισομορφικός με το χώρο διαγωνίων τελεστών $\mathcal{L}_{\rm diag}(X)$. Σε αυτή την περίπτωση $(X_*)^{**} = X^* = \mathcal{L}_{\rm diag}(X)$ με την άλγεβρα αυτή να είναι ισόμορφη της $Z^* \oplus X_*$ ή της $Z^* \oplus \mathrm{span}\{\chi_{\mathbb{N}}\} \oplus X_*$.

Σκιαγράφηση της απόδειξης. Θεωρούμε $\mathcal{L} = (\Lambda_i)_{i\in\mathbb{N}}$ μια διαμέριση του $\mathbb{N}$ σε άπειρα σύνολα. Ορίζουμε

$$\begin{aligned} G_Z^{\mathcal{L}} &= \{\sum_{i=1}^{d} a_i \chi_{E\cap\Lambda_i},\ a_i \in \mathbb{Q},\ i=1,\ldots,d \\ &\qquad E \text{ πεπ. διάστημα του } \mathbb{N},\ \|\sum_{i=1}^{d} a_i z_i^*\| \leq 1\} \end{aligned}$$

και $G_{I,Z}^{\mathcal{L}} = G_Z^{\mathcal{L}} \cup \{\pm\chi_E :\ E \text{ πεπ. διάστημα του } \mathbb{N}\}$. Τα σύνολα αυτά είναι κλειστά στα κατά σημείο γινόμενα.

Θεωρούμε τους χώρους $Y_Z = \overline{(c_{00}(\mathbb{N}), \|\ \|_{G_Z^{\mathcal{L}}})}$ και $Y_{I,Z} = \overline{(c_{00}(\mathbb{N}), \|\ \|_{G_{I,Z}^{\mathcal{L}}})}$. Αποδεικνύεται ότι ο πρώτος δεν περιέχει τον $\ell_1(\mathbb{N})$ ενώ αν η βάση $(z_i)_{i\in\mathbb{N}}$ είναι υποσυμμετρική τότε ούτε ο δεύτερος περιέχει τον $\ell_1(\mathbb{N})$. Χρησιμοποιούμε στα παρακάτω τα σύμβολα G και Y για να συμβολίσουμε είτε τα G_Z και Y_Z ή τα $G_{I,Z}$ και $Y_{I,Z}$ αντίστοιχα.

Από ένα αποτέλεσμα του Bourgain προκύπτει ότι υπάρχει διατακτικός $\xi < \omega_1$ ώστε ο Y δεν περιέχει ℓ_1^{ξ} spreading model. Για αυτό το ξ αποδεικνύεται ότι το G είναι $\mathcal{S}_\xi$ bounded (βλ. [5]). Επιλέγοντας κατάλληλη γνησίως αύξουσα ακολουθία αριθμησίμων διατακτικών $(\xi_j)_{j\in\mathbb{N}}$ με $\xi_1 = \xi$ και κατάλληλη γνησίως αύξουσα ακολουθία φυσικών $(m_j)_{j\in\mathbb{N}}$, ο χώρος X ορίζεται να είναι η πλήρωση του χώρου $(c_{00}(\mathbb{N}), \|\ \|_D)$, όπου το σύνολο D είναι το ελάχιστο υποσύνολο του $c_{00}(\mathbb{N})$ για το οποίο

(i) $G \subset D$.

(ii) Το D είναι κλειστό στις $(\mathcal{S}_{\xi_{2j}}, \frac{1}{m_{2j}})$ πράξεις.

(iii) Το D είναι κλειστό στις $(\mathcal{S}_{\xi_{2j-1}}, \frac{1}{m_{2j-1}})$ πράξεις πάνω σε $2j-1$ special ακολουθίες.

(iv) Το D είναι κλειστό στους ρητούς κυρτούς συνδυασμούς.

(v) Το D είναι κλειστό στα κατά σημείο γινόμενα.

(vi) Το D είναι κλειστό στους περιορισμούς σε διαστήματα του $\mathbb{N}$.

Έχοντας ορίσει τον X με σημείο έναρξης το $G = G_Z^{\mathcal{L}}$, αποδεικνύεται ότι η αντιστοίχηση $Q(e_n) = z_i$ όταν $n \in \Lambda_i$ επεκτείνεται σε ένα φραγμένο γραμμικό τελεστή από το X επί του Z και ο X^* είναι ισόμορφος με τον $X_* \oplus Z^*$. Ο X και ο X_* αποδεικνύεται ότι είναι H.I, ενώ λόγω του το norming σύνολο D είναι κλειστό στα κατά σημείο γινόμενα επάγει στον X_* και στον X^* δομή Banach άλγεβρας.

Στην περίπτωση που η βάση $(z_i)_{i\in\mathbb{N}}$ είναι υποσυμμετρική και έχοντας ξεκινήσει την κατασκευή από το $G = G_{I,Z}^{\mathcal{L}}$, τότε το norming σύνολο D θα περιέχει τα συναρτησοειδή $\pm\sum_{i=1}^{n} z_i^*$ για κάθε $n \in \mathbb{N}$. Έτσι σύμφωνα με το θεώρημα του [2] που προαναφέραμε, ο X^* είναι ισομορφικός με το χώρο διαγωνίων τελεστών $\mathcal{L}_{\rm diag}(X)$. Έτσι, σε αυτή την περίπτωση $(X_*)^{**} = X^* = \mathcal{L}_{\rm diag}(X)$ και η άλγεβρα αυτή είναι ισόμορφη της $Z^* \oplus X_*$ είτε της $Z^* \oplus \mathrm{span}\{\chi_{\mathbb{N}}\} \oplus X_*$. Η διάκριση των δύο αυτών περιπτώσεων οφείλεται στο εξής γεγονός. Λόγω της κατάσκευής του ο X^* περιέχει το $\chi_{\mathbb{N}} = w^* - \sum_{i=1}^{\infty} z_i^*$. Από την άλλη ο Z^* ενδέχεται να το περιέχει (πρώτη περίπτωση) ή να μην το περιέχει (δεύτερη περίπτωση). □

Αναφορές

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Ισοπλευρικά σύνολα σε απειροδιάστατους χώρους Banach

Γιώργος Βασιλειάδης

Ορισμός 1. Έστω χώρος $(X, \|\cdot\|)$ με νόρμα και $\lambda > 0$. Ένα $S \subseteq X$ ονομάζεται λ-ισοπλευρικό σύνολο αν είναι $\|x - y\| = \lambda\ \forall x, y \in S, x \neq y$.

Σε χώρους με νόρμα πεπερασμένης διάστασης έχει δοθεί ο ακόλουθος ορισμός, τον οποίο και γενικεύουμε:

Ορισμός 2. Έστω $(X, \|\cdot\|)$ χώρος με νόρμα. Ένα $S \subseteq X$ ονομάζεται <u>αντιποδικό</u> αν $\forall x, y \in S, x \neq y$ υπάρχει $f \in X^* : f(x) < f(y)$ και $f(x) \leq f(z) \leq f(y) \forall z \in S$.

Είναι γνωστό από τους Danzer και Grünbaum ότι η μέγιστη πληθικότητα ενός αντιποδικού συνόλου S στον $\mathbb{R}^n$ είναι 2^n και αυτή επιτυγχάνεται μόνο στην περίπτωση όπου το S είναι το σύνολο των κορυφών ενός n-διάστατου παραλληλότοπου.

Το αποτέλεσμα που ακολουθεί αποδείχθηκε από το (C.M.Petty) για χώρους πεπερασμένης διάστασης:

Πρόταση. *Έστω $(X, \|\cdot\|)$ χώρος με νόρμα και $S \subseteq X$ ισοπλευρικό σύνολο. Τότε το S είναι <u>αντιποδικό</u>.*

Απόδειξη. Έστω $x, y \in S, x \neq y$. Υποθέτουμε ότι το S είναι λ-ισοπλευρικό. Από το Θεώρημα Hahn-Banach υπάρχει $f \in X^*, \|f\| = 1$ ώστε

$$f(y - x) = \|y - x\| = \lambda > 0.$$

Τότε $f(x) < f(y)$ και $f(y) = sup\{f(z) : z \in B(x, \lambda)\}$. Άρα το f είναι συναρτησοειδές στήριξης της $B(x, \lambda)$ στο y και $f(z) \leq f(y) \forall z \in S$.
Επίσης για $g = -f$ έχουμε

$$g(x - y) = f(y - x) = \|y - x\| > 0$$

και $\|g\| = 1$, οπότε παρόμοια το g είναι συναρτησοειδές στήριξης της $B(y, \lambda)$ στο x και $g(z) \leq g(x) \, \forall z \in S$.
Άρα $f(x) \leq f(z) \leq f(y) \, \forall z \in S$ και το σύνολο S είναι αντιποδικό. $\square$

Θεώρημα. *(C.M.Petty)*
Έστω $(X, \|\cdot\|)$ χώρος με νόρμα πεπερασμένης διάστασης και $S \subseteq X$ αντιποδικό σύνολο. Τότε υπάρχει ισοδύναμη νόρμα $|||\cdot|||$ στον X, ώστε το S να είναι ισοπλευρικό σύνολο στον $(X, |||\cdot|||)$.

Θα γενικεύσουμε το Θεώρημα του C.M.Petty σε απειροδιάστατους χώρους.

Ορισμός 3. Έστω $(X, \|\cdot\|)$ χώρος με νόρμα. Ένα $S \subseteq X$ ονομάζεται φραγμένο και διαχωρισμένο αντιποδικό σύνολο, αν υπάρχουν σταθερές $c, d > 0$ ώστε:
(i) $S \subseteq B(0, c)$
(ii) $\forall x, y \in S, x \neq y$ υπάρχει $f \in B_{X^*}$ ώστε $0 < d \leq f(y) - f(x)$ και $f(x) \leq f(z) \leq f(y) \, \forall z \in S$.

Παρατηρήσεις.

1. Κάθε ισοπλευρικό σύνολο σε χώρο με νόρμα είναι φραγμένο και διαχωρισμένο αντιποδικό σύνολο.

2. Ένα διορθογώνιο σύστημα $(x_\gamma, {x_\gamma}^*) \in X \times X^*$ σε χώρο με νόρμα δίνει αντιποδικό σύνολο.
Πράγματι, τότε έχουμε:
$$ {x_{\gamma_1}}^*(x_{\gamma_2}) = \delta_{\gamma_1 \gamma_2} \, \forall \gamma_1, \gamma_2 \in \Gamma. $$
Αν $\gamma_1 \neq \gamma_2$,είναι
$$ 0 = {x_{\gamma_1}}^*(x_{\gamma_2}) \leq {x_{\gamma_1}}^*(x_\gamma) \leq {x_{\gamma_1}}^*(x_{\gamma_1}) = 1 \, \forall \gamma \in \Gamma. $$

3. Έστω $\{x_\gamma; f_\gamma\}$ ένα φραγμένο διορθογώνιο σύστημα στον X, δηλ. υπάρχει $M > 0$ ώστε
$$ \|x_\gamma\| \cdot \|f_\gamma\| \leq M \, \forall \gamma \in \Gamma. $$
Μπορώ τότε να θέσω $g_\gamma = \frac{f_\gamma}{\|f_\gamma\|}, \gamma \in \Gamma$ και $y_\gamma = x_\gamma \cdot \|f_\gamma\|, \gamma \in \Gamma$ οπότε το διορθογώνιο σύστημα $\{y_\gamma; g_\gamma\}$ δίνει $\{y_\gamma\}_{\gamma \in \Gamma}$ φραγμένο και διαχωρισμένο αντιποδικό σύνολο.

Θεώρημα. *Έστω $(X, \|\cdot\|)$ χώρος με νόρμα και $S \subseteq X$ φραγμένο και διαχωρισμένο αντιποδικό σύνολο. Τότε υπάρχει ισοδύναμη νόρμα $|||\cdot|||$ στον X, ώστε το S να είναι ισοπλευρικό σύνολο στον $(X, |||\cdot|||)$.*
(Αν οι σταθερές του S είναι c, d τότε η Banach-Mazur απόσταση των δύο νορμών είναι $\leq \frac{2c}{d}$).

Απόδειξη. Θέτουμε

$$K = \overline{conv}(d \cdot B_x \cup \{x - y : x, y \in S\}).$$

Το K είναι κλειστό (φραγμένο), κυρτό και συμμετρικό σύνολο με $0 \in int(K)$, επομένως το συναρτησοειδές Minkowski ορίζει μια νόρμα

$$\|x\|_K = inf\{\lambda > 0 : x \in \lambda K\}$$

και η μπάλα του χώρου $(X, \|\cdot\|_K)$ είναι ακριβώς το σύνολο K. Για $x, y \in S, x \neq y$ υπάρχει $f \in B_{X^*}$ ώστε:

$$d \leq f(y) - f(x) \leq \|f\|\|x - y\| \leq 2c$$

επομένως $d \cdot B_X \subseteq K \subseteq 2c \cdot B_X$ και έπεται ότι η Banach-Mazur απόσταση των δύο νορμών είναι $\leq \frac{2c}{d}$.
Αρκεί τώρα να δείξουμε ότι αν $x, y \in S$ με $x \neq y$ τότε $x - y \in \partial K \Leftrightarrow \|x - y\|_K = 1$ (επομένως το S είναι 1-ισοπλευρικό στη $\|\cdot\|_K$).
Έστω λοιπόν $x, y \in S$ με $x \neq y$. Τότε υπάρχει $f \in B_{X^*}$ με $d \leq f(y) - f(x)$ και $f(x) \leq f(z) \leq f(y) \forall z \in S$.
Τότε για $z_1, z_2 \in S$ είναι

$$f(z_1 - z_2) \leq f(y - x).$$

Επίσης αν $z \in d \cdot B_X$, τότε

$$f(z) \leq |f(z)| \leq \|z\| \leq d \leq f(y - x)$$

και άρα το f είναι συναρτησοειδές στήριξης του K στο σημείο $y - x$, συνεπώς $y - x \in \partial K$. □

Πόρισμα. *Έστω $\{x_\gamma; f_\gamma\}$ ένα φραγμένο διορθογώνιο σύστημα στον $(X, \|\cdot\|)$ με $\|f_\gamma\| = 1, \gamma \in \Gamma$ και $\|x_\gamma\| \leq c, \gamma \in \Gamma$. Τότε υπάρχει ισοδύναμη νόρμα $|||\cdot|||$ στον X με Banach-Mazur απόσταση από την αρχική $\leq 2c$ $(c \geq 1)$, ώστε το συνολο $\{x_\gamma : \gamma \in \Gamma\}$ να είναι 1-ισοπλευρικό στον $(X, |||\cdot|||)$.*

Αυτό το απέδειξε ο K.J.Swanepoel στην περίπτωση όπου ο χώρος είναι διαχωρίσιμος, παίρνοντας ισοδύναμη νόρμα με Banach-Mazur απόσταση από την αρχική $\leq 2+\varepsilon$.
Χρησιμοποιώντας (στη διαχωρίσιμη περίπτωση) ένα αποτέλεσμα του Day για την ύπαρξη άπειρου αριθμήσιμου Auerbach συστήματος, μπορούμε να πάρουμε ισοδύναμη νόρμα με Banach-Mazur απόσταση ≤ 2 από την αρχική.

Παρατήρηση. Αν ο $(X, \|\cdot\|)$ περιέχει ισομορφικά τον ℓ^1 ή τον c_0, τότε $\forall \varepsilon > 0$ υπάρχει κλειστός υπόχωρος Y του X και ισοδύναμη νόρμα στον Y ώστε:
(i) $d((Y, |||\cdot|||), (Y, \|\cdot\|)) \leq 1+\varepsilon$ και
(ii) Ο $(Y, |||\cdot|||)$ περιέχει άπειρο ισοπλευρικό σύνολο.
Αυτό συμβαίνει λόγω της ιδιότητας της μη-παραμόρφωσης (distortion) της νόρμας του c_0 και του ℓ^1 (ο χώρος θα περιέχει αντίτυπα του c_0 ή του ℓ^1 με μικρή Banach-Mazur απόσταση από την αρχική νόρμα).

Παρατήρηση. Υπάρχουν κλάσεις μη διαχωρίσιμων χώρων Banach οι οποίοι έχουν διορθογώνια συστήματα:

1. Οι WCG χώροι και οι γενικεύσεις τους (μάλιστα έχουν M-βάσεις)
2. Οι χώροι που είναι representable (π.χ. οι δυϊκοί διαχωρίσιμων χώρων Banach, οι οποίοι δεν είναι διαχωρίσιμοι)
3. Αν ο X είναι μη διαχωρίσιμος χώρος Banach και είναι ισόμορφος με δυϊκό χώρο Banach, τότε δέχεται υπεραριθμήσιμο διορθογώνιο σύστημα.

Σημειώνουμε ότι είναι συνεπές στην ZFC να υποθέσουμε ότι κάθε μη διαχωρίσιμος χώρος Banach δέχεται υπεραριθμήσιμο διορθογώνιο σύστημα.

Έστω K συμπαγής και Hausdorff τοπολογικός χώρος. Μια συνεχής συνάρτηση $f : K \to [0,1]$ λέγεται συνάρτηση Urysohn, αν $f^{-1}(\{0\}) \neq \emptyset$ και $f^{-1}(\{1\}) \neq \emptyset$.
Αν $A, B \neq \emptyset$ κλειστά και ξένα υποσύνολα του K τότε υπάρχει (από το Λήμμα Urysohn) μια συνάρτηση Urysohn $f : K \to [0,1]$ με $f/A = 1$ και $f/B = 0$. Αν επιπλέον τα A, B είναι G_δ σύνολα, τότε υπάρχει συνάρτηση Urysohn με $f^{-1}(\{1\}) = A$ και $f^{-1}(\{0\}) = B$.

Παρατηρήσεις. 1. Ένας συμπαγής χώρος K είναι κληρονομικά Lindelöf αν και μόνο αν είναι perfectly normal (ισοδύναμα κάθε κλειστό υποσύνολό του είναι G_δ-σύνολο).

2. Έστω K συμπαγής χώρος και f, g συναρτήσεις Urysohn. Τότε είναι:
$\|f - g\|_\infty \leq 1$ και
$\|f - g\|_\infty = 1 \Leftrightarrow f^{-1}(\{1\}) \cap g^{-1}(\{0\}) \neq \emptyset$ ή $g^{-1}(\{1\}) \cap f^{-1}(\{0\}) \neq \emptyset$.

Ορίζουμε ακολούθως μια ασθενή έννοια ανεξαρτησίας:

Ορισμός 4. Έστω S ένα σύνολο και $(A_\alpha, B_\alpha)_{\alpha \in I}$ μια οικογένεια μη κενών υποσυνόλων του S ώστε:
(i) $A_\alpha \cap B_\alpha = \emptyset, \alpha \in I$ και
(ii) $\forall a, b \in I\ a \neq b$ ισχύει είτε $A_\alpha \cap B_b \neq \emptyset$ ή $A_b \cap B_\alpha \neq \emptyset$.

Λήμμα. *Έστω K συμπαγής και Hausdorff τοπολογικός χώρος και $\mathcal{F} \subseteq C(K)$ μια οικογένεια συναρτήσεων Urysohn. Τα ακόλουθα είναι ισοδύναμα:*

1. *Το σύνολο $\mathcal{F}$ είναι ισοπλευρικό στον $(C(K), \|\cdot\|_\infty)$ (δηλ. $\|f - g\|_\infty = 1 \forall f, g \in \mathcal{F}, f \neq g$).*
2. *Η οικογένεια ξένων κλειστών συνόλων $(f^{-1}(\{1\}), f^{-1}(\{0\}))$ είναι "ανεξάρτητη".*

Πρόταση. *Έστω K συμπαγής και Hausdorff τοπολογικός χώρος και $(A_\alpha, B_\alpha)_{\alpha \in I}$ μια "ανεξάρτητη' οικογένεια κλειστών υποσυνόλων του K. Θεωρούμε για $\alpha \in A$ μια συνάρτηση Urysohn $f_\alpha : K \to [0, 1]$ με $f_\alpha / A_\alpha = 1$ και $f_\alpha / B_\alpha = 0$. Τότε η οικογένεια συναρτήσεων $\{f_\alpha : \alpha \in A\}$ είναι ισοπλευρική στον $C(K)$.*

Έστω K συμπαγής και Hausdorff τοπολογικός χώρος και $\mathcal{C}$ μια οικογένεια ανοικτών-κλειστών υποσυνόλων του K με $\emptyset \neq A \neq K, \forall A \in \mathcal{C}$. Τότε η οικογένεια $\{(A, K \setminus A) : A \in \mathcal{C}\}$ είναι "ανεξάρτητη' και επομένως η οικογένεια συναρτήσεων $\{f_A = \chi_A : A \in \mathcal{C}\}$ είναι ισοπλευρική στον $C(K)$.

Πόρισμα. *Έστω K συμπαγής και Hausdorff, ολικά μη συνεκτικός χώρος. Τότε υπάρχει ισοπλευρικό σύνολο $\mathcal{F} \subseteq C(K)$ με $|\mathcal{F}| = w(K)$. ($\mathcal{F} = \{\chi_V : V \subseteq K, V \in \mathcal{B}\}$)*

Ορισμός 5. Μια οικογένεια $\{x_\alpha : \alpha < \omega_1\}$ στον τοπολογικό χώρο X ονομάζεται
<u>δεξιά διαχωρισμένη</u>, αν $x_\alpha \notin \overline{\{x_\beta : \alpha < \beta < \omega_1\}}, \forall \alpha < \omega_1$.

Πρόταση. *Ένας τοπολογικός χώρος X είναι κληρονομικά Lindelöf αν και μόνο αν ο X δεν περιέχει δεξιά διαχωρισμένη οικογένεια.*

Πρόταση. *Έστω K συμπαγής και Hausdorff τοπολογικός χώρος, ο οποίος δεν είναι κληρονομικά Lindelöf. Τότε ο $(C(K), \|\cdot\|_\infty)$ περιέχει υπεραριθμήσιμο ισοπλευρικό σύνολο.*

Απόδειξη. Έστω $\{t_\alpha : \alpha < \omega_1\} \subseteq K$ μια δεξιά διαχωρισμένη οικογένεια. Θέτουμε

$$H_\alpha = \{t_\alpha\} \; F_\alpha = \overline{\{t_\beta : \alpha < \beta < \omega_1\}}$$

με $H_\alpha \cap F_\alpha = \emptyset, \forall \alpha < \omega_1$ και για $\alpha < \beta < \omega_1$ είναι $t_\beta \in F_\alpha$. Η οικογένεια $(H_\alpha, F_\alpha)_{\alpha<\omega_1}$ είναι "ανεξάρτητη".

□

Πρόταση. *Έστω K συμπαγής και Hausdorff τοπολογικός χώρος, ο οποίος δεν είναι κληρονομικά διαχωρίσιμος. Τότε ο $C(K)$ περιέχει υπεραριθμήσιμο ισοπλευρικό σύνολο.*

Απόδειξη. Υπάρχει μια οικογένεια $(t_\alpha)_{\alpha<\omega_1}$ αριστερά διαχωρισμένη και θέτουμε:
$F_\alpha = \overline{\{t_\beta : \beta < \alpha\}}, \alpha < \omega_1$ κλειστό, μη κενό με $t_\beta \in F_\alpha$, $\beta < \alpha$
και η οικογένεια $(\{t_\alpha\}, F_\alpha)_{\alpha<\omega_1}$ είναι "ανεξάρτητη".

□

Πρόταση. *Αν X μη διαχωρίσιμος χώρος, τότε η μπάλα του X^* (B_{X^*}, w^*) δεν είναι κληρονομικά Lindelöf και επομένως ο χώρος Banach $C(B_{X^*})$ περιέχει υπεραριθμήσιμο ισοπλευρικό σύνολο.*

Ερωτήματα:

1. Έστω ο χώρος $(C(K), \|\cdot\|_\infty)$, K συμπαγής και μη μετρικοποιήσιμος. Υπάρχει $S \subseteq C(K)$ ισοπλευρικό σύνολο στη $\|\cdot\|_\infty$;

2. Υπάρχει χώρος $(X, \|\cdot\|_\infty)$ μη διαχωρίσιμος, ώστε κάθε αντιποδικό, φραγμένο και διαχωρισμένο σύνολο είναι το πολύ αριθμήσιμο;